

DEFORMATIONS OF LEVI FLAT STRUCTURES IN SMOOTH MANIFOLDS

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ABSTRACT. We study intrinsic deformations of Levi flat structures on a smooth manifold. A Levi flat structure on a smooth manifold L is a couple (ξ, J) where $\xi \subset T(L)$ is an integrable distribution of codimension 1 and $J : \xi \rightarrow \xi$ is a bundle automorphism which defines a complex integrable structure on each leaf. A deformation of a Levi flat structure (ξ, J) is a smooth family $\{(\xi_t, J_t)\}_{t \in]-\varepsilon, \varepsilon[}$ of Levi flat structures on L such that $(\xi_0, J_0) = (\xi, J)$. We define a complex whose cohomology group of order 1 contains the infinitesimal deformations of a Levi flat structure. In the case of real analytic Levi flat structures, this cohomology group is $H^1(Z^*(L), \delta) \times H^1(\Lambda_J^{0,*}(\xi) \otimes \xi, \bar{\partial}_J)$ where $(Z^*(L), \delta, \{\cdot, \cdot\})$ is the DGLA associated to ξ .

1. INTRODUCTION

Let Ω be a domain with C^2 boundary in \mathbb{C}^n , $n \geq 2$, $\Omega = \{z \in U : \rho(z) < 0\}$ where ρ is a C^2 function defined in a neighborhood U of $\partial\Omega$ such that $d\rho \neq 0$ on $\partial\Omega$. The Levi form introduced by E. E. Levi in [11] is the Hermitian form

$$\mathcal{L}_\rho(z, w) = \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(z) w_i \bar{w}_j, \quad z \in \partial\Omega, \quad w \in T_z^\mathbb{C}(\partial\Omega)$$

where $T_z^\mathbb{C}(\partial\Omega) = T_z(\partial\Omega) \cap JT_z(\partial\Omega)$ is the maximal complex subspace contained in the tangent space $T_z(\partial\Omega)$ at z to $\partial\Omega$ and J is the standard complex structure of \mathbb{C}^n . The semipositivity of the Levi form characterizes the domains of holomorphy of \mathbb{C}^n ([17], [1], [16]).

A special situation occurs when the Levi form vanishes i.e. $\partial\Omega$ is Levi flat. This case is related to a foliation of $\partial\Omega$ by complex hypersurfaces, as it was firstly remarked by E. Cartan in [2]. In fact the following theorem is implicit in [2]: a real analytic hypersurface in \mathbb{C}^n is Levi flat if and only if it is locally biholomorphic to a real hyperplane in \mathbb{C}^n . This result was generalized for smooth hypersurfaces by F. Sommer [18]: a smooth real hypersurface L in a complex manifold M is Levi flat if and only if the distribution $\xi = TL \cap JTL$ is integrable, where J is the complex structure of M .

These notions have an intrinsic equivalent: a Levi flat structure on a smooth manifold L is a couple (ξ, J) where $\xi \subset T(L)$ is an integrable distribution of

Date: May, 22, 2012.

1991 *Mathematics Subject Classification.* Primary 32G05, 32G07, 32G08, 32G10; Secondary, 17B70, 51M99, 32Q99, 58A30.

Key words and phrases. Levi flat structures, Differential Graduate Lie Algebras, Maurer-Cartan equation, Complex Lie algebras of derivation type, Nijenhuis tensor .

codimension 1 and $J : \xi \rightarrow \xi$ is a bundle automorphism which defines a complex integrable structure on each leaf.

It was proved by W. Lickorish in [12] (and in an unpublished paper by S. Novikov and H. Zieschang) that any compact orientable 3-manifold has a foliation of codimension 1. J. Wood proved in [20] that any compact 3-manifold has a transversally orientable foliation of codimension 1. It follows that any compact orientable 3-manifold admits a Levi flat structure. In upper dimensions the situation is more complicated and it seems that the problem of the existence of a Levi flat structure on \mathbb{S}^{2n+1} , $n \geq 2$, is still open (see [13] and [14]).

In this paper we study the deformations of Levi flat structures. The theory of deformations of complex manifolds was intensively studied from the 50s beginning with the famous results of Kodaira and Spencer [8] (see for ex. [7], [19]). In [15], Nijenhuis and Richardson proved that the deformations of complex structures are given by solutions of the Maurer-Cartan equation in a graded Lie algebra by using a theory initiated by Gerstenhaber [5]. This theory was developed following ideas of Deligne by Goldman and Millson [6] and in more general situations by M. Kontsevich [9], [10].

It is thus interesting to investigate the deformation theory of other structures involving complex manifolds via the Maurer-Cartan equation in an adapted DGLA. This was done by de Bartolomeis and Meylan [4] for strictly pseudoconvex CR structures of hypersurface type on a contact manifold.

In [3] the authors studied deformations of Levi flat hypersurfaces L in compact complex manifolds M . In this case the complex structure on the leaves is induced by the complex structure of M .

As the Levi flat hypersurfaces are characterized by the integrability of the Levi distribution, we studied firstly in [3] deformations of integrable distributions of codimension 1 on smooth manifolds. Thus, we defined a Differential Graded Lie Algebra (DGLA) $(\mathcal{Z}^*(L), \delta, \{\cdot, \cdot\})$ associated to an integrable distribution of codimension 1 such that the deformations of this distribution are given by solutions of Maurer-Cartan equation in this algebra. Then we considered a smooth Levi flat hypersurface L in a complex manifold and we gave a parametrization of families of smooth hypersurfaces near L such that the Levi flat deformations are given by the solutions of the Maurer-Cartan equation in the DGLA associated to the Levi foliation. Here a Levi-flat deformation of L is a smooth application $\Psi : I \times M \rightarrow M$, where I is an interval in \mathbb{R} containing the origin, such that $\Psi_t = \Psi(t, \cdot) \in \text{Diff}_0(M)$, $L_t = \Psi_t L$ is a Levi flat hypersurface in M for every $t \in I$ and $L_0 = L$. This allowed us to characterize the infinitesimal deformations of Levi-flat hypersurfaces in a complex manifold.

The purpose of this paper is to study intrinsic deformations of Levi flat structures on a smooth manifold. Let L be a smooth manifold and (ξ, J) a Levi flat structure on L . A deformation of a Levi flat structure (ξ, J) is a smooth family $\{(\xi_t, J_t)\}_{t \in]-\varepsilon, \varepsilon[}$ of Levi flat structures on L such that $(\xi_0, J_0) = (\xi, J)$.

This situation is totally different from the case of deformations of Levi flat hypersurfaces in complex manifolds, since the complex structure of the leaves is not fixed.

In the first paragraph we recall the results obtained in [3] for the deformations of integrable distribution of codimension 1 and we give a few proofs which are used in the sequel.

Then we introduce the complex Lie algebras of derivation type with their Nijenhuis tensor and $\bar{\partial}$ -operator. A Levi flat structure (ξ, J) on a smooth manifold L induces a natural complex Lie algebra of derivation type on the algebra $\mathcal{H}(\xi)$ of the vector fields on L which are tangent to ξ .

We define a $(0, 1)$ -form $H_{J, \gamma, X}$ associated to a DGLA defining couple (γ, X) whose 1-cohomology class in a modified $\bar{\partial}_J$ -complex $\bar{\partial}_{J, \gamma, X} : \Lambda^{0, *}(\xi) \otimes \xi \rightarrow \Lambda^{0, *}(\xi) \otimes \xi$ is independent on the choice of (γ, X) . If the cohomology class of $H_{J, \gamma, X}$ in this complex vanishes, the Levi flat structure is called exact. Every real analytic Levi flat hypersurface in a complex manifold is exact.

To study infinitesimal deformations of a Levi flat structure (ξ, J) , we define a complex $\mathfrak{Z}^*(L, \xi)$ whose cohomology group of order 1 contains the set of infinitesimal deformations of (ξ, J) . If this set reduces to a point and thus in particular if the first cohomology group vanishes, we say that (ξ, J) is infinitesimally rigid.

If (ξ, J) is exact, then $H^1(\mathfrak{Z}, \mathfrak{d}) = H^1(\mathcal{Z}(L), \delta) \times H^1(\Lambda_J^{0, *}(\xi) \otimes \xi, \bar{\partial}_J)$.

Some proofs require tedious computations and we tried to make them as easy as possible to read.

2. DEFORMATION THEORY OF INTEGRABLE DISTRIBUTION OF CODIMENSION 1

For simplicity, all the objects considered in the sequel will be smooth of class C^∞ . For the convenience of the reader we recall in this paragraph several basic definitions and results from [3]:

2.1. DGLA defining couples.

Definition 1. A differential graded Lie algebra (DGLA) is a triple $(V^*, d, [\cdot, \cdot])$ such that:

- 1) $V^* = \bigoplus_{i \in \mathbb{N}} V^i$, where $(V^i)_{i \in \mathbb{N}}$ is a family of \mathbb{C} -vector spaces and $d : V^* \rightarrow V^*$ is a graded homomorphism such that $d^2 = 0$. An element $a \in V^k$ is said to be homogeneous of degree $k = \deg a$.
- 2) $[\cdot, \cdot] : V^* \times V^* \rightarrow V^*$ defines a structure of graded Lie algebra i.e. for homogeneous elements we have

$$(2.1) \quad [a, b] = -(-1)^{\deg a \deg b} [b, a]$$

and

$$(2.2) \quad [a, [b, c]] = [[a, b], c] + (-1)^{\deg a \deg b} [b, [a, c]]$$

- 3) d is compatible with the graded Lie algebra structure i.e.

$$(2.3) \quad d[a, b] = [da, b] + (-1)^{\deg a} [a, db].$$

Definition 2. Let $(V^*, d, [\cdot, \cdot])$ be a DGLA and $a \in V^1$. We say that a verifies the Maurer-Cartan equation in $(V^*, d, [\cdot, \cdot])$ if

$$(2.4) \quad da + \frac{1}{2} [a, a] = 0.$$

Definition 3. Let L be a C^∞ manifold and $\xi \subset T(L)$ an integrable distribution of codimension 1. A couple (γ, X) where $\gamma \in \Lambda^1(L)$ and X is a vector field on L such that $\text{Ker } \gamma = \xi$ and $\gamma(X) = 1$ will be called a DGLA defining couple.

Lemma 1. *Let L be a C^∞ manifold and X a vector field on L . We denote by $\Lambda^k(L)$ the k -forms on L and $\Lambda^*(L) = \bigoplus_{k \in \mathbb{N}} \Lambda^k(L)$. For $\alpha, \beta \in \Lambda^*(L)$, set*

$$(2.5) \quad \{\alpha, \beta\} = \mathcal{L}_X \alpha \wedge \beta - \alpha \wedge \mathcal{L}_X \beta$$

where \mathcal{L}_X is the Lie derivative. Then $(\Lambda^*(L), d, \{\cdot, \cdot\})$ is a DGLA.

Lemma 2. *Let L be a C^∞ manifold and $\xi \subset T(L)$ a distribution of codimension*

1. *Let (γ, X) be a DGLA defining couple. Then the following are equivalent:*

- i) ξ is integrable;
- ii) There exists $\alpha \in \Lambda^1(L)$ such that $d\gamma = \alpha \wedge \gamma$;
- iii) $d\gamma \wedge \gamma = 0$;
- iv) $d\gamma = -\iota_X d\gamma \wedge \gamma$;
- v) γ satisfies the Maurer-Cartan equation (2.4) in $(\Lambda^*(L), d, \{\cdot, \cdot\})$, where $\{\cdot, \cdot\}$ is defined in (2.5).

Corollary 1. *Let L be a C^∞ manifold and $\xi \subset T(L)$ an integrable distribution of codimension 1. Let (γ, X) be a DGLA defining couple. Set*

$$\delta = d_\gamma = d + \{\gamma, \cdot\}$$

where $\{\cdot, \cdot\}$ is defined in (2.5). Then $(\Lambda^*(L), \delta, \{\cdot, \cdot\})$ is a DGLA.

Corollary 2. *Under the hypothesis of Corollary 1, we set*

$$\mathcal{Z}^*(L) = \{\alpha \in \Lambda^*(L) : \iota_X \alpha = 0\}.$$

Then $(\mathcal{Z}^*(L), \delta, \{\cdot, \cdot\})$ is a sub-DGLA of $(\Lambda^*(L), \delta, \{\cdot, \cdot\})$.

Remark 1. *Let $\alpha, \beta \in \mathcal{Z}^*(L)$ and (γ, X) a DGLA defining couple. Then*

$$(2.6) \quad \{\alpha, \beta\} = (\iota_X d + d\iota_X) \alpha \wedge \beta - \alpha \wedge (\iota_X d + d\iota_X) \beta = \iota_X d\alpha \wedge \beta - \alpha \wedge \iota_X d\beta$$

and

$$(2.7) \quad \{\gamma, \alpha\} = (\iota_X d + d\iota_X) \gamma \wedge \alpha - \gamma \wedge (\iota_X d + d\iota_X) \alpha = \iota_X d\gamma \wedge \alpha - \gamma \wedge \iota_X d\alpha.$$

Let L be a C^∞ manifold and $\xi \subset T(L)$ an integrable distribution of codimension 1. We fix a DGLA defining couple (γ, X) and we consider the DGLA $(\mathcal{Z}^*(L), \delta, \{\cdot, \cdot\})$ previously defined.

Lemma 3. *Let $\alpha \in \mathcal{Z}^1(L)$. The following are equivalent:*

- i) The distribution $\xi_\alpha = \ker(\gamma + \alpha)$ is integrable.
- ii) α satisfies the Maurer-Cartan equation (2.4) in $(\mathcal{Z}^*(L), \delta, \{\cdot, \cdot\})$.

2.2. Group action.

Definition 4. *Let $\mathcal{G} = \text{Diff}(L)$ be the group of diffeomorphisms of L . Let \mathcal{U}_0 be a neighborhood of the identity in \mathcal{G} and \mathcal{V}_0 be a neighborhood of 0 in $\mathcal{Z}^1(L)$ such that $\Phi^*(\gamma + \alpha)(X) \neq 0$ for every $\Phi \in \mathcal{U}_0$ and every $\alpha \in \mathcal{V}_0$. We define*

$$(2.8) \quad (\Phi, \alpha) \in \mathcal{U}_0 \times \mathcal{V}_0 \subset \mathcal{G} \times \mathcal{Z}^1(L) \rightarrow \mathcal{Z}^1(L) \ni \chi(\Phi)(\alpha) = (\Phi^*(\gamma + \alpha)(\mathfrak{X}))^{-1} \Phi^*(\gamma + \alpha) - \gamma.$$

Remark 2. *In this way, $\xi_{\chi(\Phi)(\alpha)} = \Phi^* \xi_\alpha$. This means that ξ_α is integrable if and only if $\xi_{\chi(\Phi)(\alpha)}$ is integrable. By Lemma 3 we deduce that α satisfies the Maurer-Cartan equation (2.4) in the DGLA $(\mathcal{Z}^*(L), \delta, \{\cdot, \cdot\})$ if and only if $\chi(\Phi)(\alpha)$ does.*

So we obtain the following

Proposition 1. *Set*

$$\mathfrak{M}\mathfrak{C}_\delta(L) = \left\{ \alpha \in \mathcal{Z}^1(L) : \delta a + \frac{1}{2} \{\alpha, \alpha\} = 0 \right\}.$$

Then the moduli space of deformations of integrable distributions of codimension 1 is

$$\mathfrak{M}\mathfrak{C}_\delta(L) / \sim_{\mathcal{G}}$$

where $\alpha \sim_{\mathcal{G}} \beta$ if there exists $\Phi \in \mathcal{G}$ such that $\beta = \chi(\Phi)(\alpha)$.

Lemma 4. *Let Y be a vector field on L and Φ^Y the flow of Y . Then*

$$\frac{d\chi(\Phi_t^Y)}{dt} \Big|_{t=0} (0) = -\delta(\iota_Y \gamma).$$

Proof. We have

$$\begin{aligned} \frac{d\chi(\Phi_t^Y)}{dt} \Big|_{t=0} (0) &= \frac{d \left(\left(\left((\Phi_t^Y)^{-1} \right)^* (\gamma)(X) \right)^{-1} \left((\Phi_t^Y)^{-1} \right)^* (\gamma) - \gamma \right)}{dt} \Big|_{t=0} \\ &= \left((\Phi_t^Y)^{-1} \right)^* (\gamma) \frac{d \left(\left((\Phi_t^Y)^{-1} \right)^* (\gamma)(X)^{-1} \right)}{dt} \Big|_{t=0} (0) \\ &\quad + \left((\Phi_t^Y)^{-1} \right)^* (\gamma)(X)^{-1} \frac{d \left((\Phi_t^Y)^{-1} \right)^*}{dt} \Big|_{t=0} \\ &= \frac{d \left(\left(\left((\Phi_t^Y)^{-1} \right)^* (\gamma)(X) \right)^{-1} \right)}{dt} \Big|_{t=0} \gamma + \frac{d \left(\left((\Phi_t^Y)^{-1} \right)^* (\gamma) \right)}{dt} \Big|_{t=0} \\ &= \mathcal{L}_Y(\gamma)(X) \gamma - \mathcal{L}_Y \gamma \\ &= (d\iota_Y \gamma)(X) \gamma + \iota_Y d\gamma(X) \gamma - d\iota_Y \gamma - \iota_Y d\gamma. \end{aligned}$$

By Lemma 2 iv)

$$\begin{aligned} \iota_Y d\gamma &= -\iota_Y (\iota_X d\gamma \wedge \gamma) = -(\iota_Y (\iota_X d\gamma)) \gamma + (\iota_Y \gamma) \iota_X d\gamma \\ &= -(d\gamma(X, Y)) \gamma + (\iota_Y \gamma) \iota_X d\gamma, \end{aligned}$$

so

$$\begin{aligned} \frac{d\chi(\Phi_t^Y)}{dt} \Big|_{t=0} (0) &= (d\iota_Y \gamma)(X) \gamma - d\gamma(Y, X) \gamma - d\iota_Y \gamma \\ &\quad + (d\gamma(X, Y)) \gamma - (\iota_Y \gamma) \iota_X d\gamma \\ (2.9) \quad &= (\iota_X d\iota_Y \gamma) \gamma - d\iota_Y \gamma - (\iota_Y \gamma) \iota_X d\gamma. \end{aligned}$$

Since

$$\mathcal{L}_X \gamma = d\iota_X \gamma + \iota_X d\gamma = \iota_X d\gamma$$

it follows that

$$\begin{aligned} (2.10) \quad \delta\iota_Y \gamma &= d\iota_Y \gamma + \{\gamma, \iota_Y \gamma\} = d\iota_Y \gamma + \mathcal{L}_X \gamma \wedge \iota_Y \gamma - \gamma \wedge \mathcal{L}_X \iota_Y \gamma \\ &= d\iota_Y \gamma + (\iota_Y \gamma) \iota_X d\gamma - X(\iota_Y \gamma) \gamma. \end{aligned}$$

From (2.9) and (2.10) we obtain

$$\frac{d\chi(\Phi_t^Y)}{dt} \Big|_{t=0} (0) = -\delta\iota_Y \gamma.$$

□

2.3. Infinitesimal deformations.

Definition 5. A $\mathfrak{MC}_\delta(L)$ -valued curve through the origin is a smooth mapping $\lambda : [-a, a] \rightarrow \mathfrak{MC}_\delta(L)$, $a > 0$, such that $\lambda(0) = 0$. We say that α is the tangent vector at the origin of the $\mathfrak{MC}_\delta(L)$ -valued curve λ through the origin to $\mathfrak{MC}_\delta(L)$ if $\alpha = \lim_{t \rightarrow 0} \frac{\lambda(t)}{t} = \frac{d\lambda}{dt} \big|_{t=0}$.

Proposition 2. Let α be the tangent vector at the origin of a $\mathfrak{MC}_\delta(L)$ -valued curve through the origin λ , Y a vector field on L and Φ^Y the flow of Y . Set $\mu(t) = \chi(\Phi_t^Y)(\lambda(t))$. Then:

i) $\delta\alpha = 0$.

ii) The tangent vector β at the origin of the $\mathfrak{MC}_\delta(L)$ -valued curve μ is

$$\beta = \alpha - \delta\iota_Y \gamma$$

Proof. i) By Lemma 3 $\lambda(t)$ verifies the Maurer Cartan equation for every t . Since $\lambda(t) = \alpha t + o(t)$, we have $\delta\alpha = 0$.

ii)

$$(2.11) \quad \beta = \frac{d\mu}{dt} \big|_{t=0} = \frac{d}{dt} \chi(\Phi_t^Y(\lambda(t))) \big|_{t=0} = \frac{d\chi(\Phi_t^Y)}{dt} \big|_{t=0} (0) + \alpha.$$

The Proposition 2 follows now by Lemma 4. □

The Proposition 2 justifies the following definition:

Definition 6. The infinitesimal deformations of ξ is the collection of cohomology classes in $H^1(\mathcal{Z}(L), \delta)$ of the tangent vectors at 0 to $\mathfrak{MC}_\delta(L)$ -valued curves. We denote by $T_{[0]}(\mathfrak{MC}_\delta(L) / \sim_{\mathcal{G}})$ the set of infinitesimal deformations of ξ .

2.4. d_b operator.

Remark 3. We denote $\Lambda^*(\xi) = \bigoplus_{p \in \mathbb{N}} \Lambda^p \xi^*$. There exists a natural isomorphism $\Theta : \Lambda^*(\xi) \rightarrow \mathcal{Z}^*(L)$: for $\alpha \in \Lambda^1 \xi^*$ set $\Theta(\alpha)(X) = 0$, $\Theta(\alpha)(Y) = \alpha(Y)$ if $Y \in \xi$ and extend by linearity. Let $d_b : \Lambda^*(\xi) \rightarrow \Lambda^*(\xi)$ be the differential along the leaves of ξ . By using this isomorphism we consider $d_b : \mathcal{Z}^*(L) \rightarrow \mathcal{Z}^*(L)$ and for every $\alpha \in \mathcal{Z}^*(L)$ we have

$$(2.12) \quad d_b \alpha = \iota_X (\gamma \wedge d\alpha) = d\alpha - \gamma \wedge \iota_X d\alpha.$$

Indeed let $\alpha \in \Lambda^p \xi^*$ and $X_1, \dots, X_{p+1} \in \xi$. Since $\gamma(X_j) = 0$, $j = 1, \dots, p+1$ and $\gamma(X) = 1$, we have

$$\iota_X (\gamma \wedge d\alpha)(X_1, \dots, X_{p+1}) = (\gamma \wedge d\alpha)(X, X_1, \dots, X_{p+1}) = d\alpha(X_1, \dots, X_{p+1}).$$

Lemma 5. The form $\iota_X d\gamma$ is d_b -closed.

3. COMPLEX LIE ALGEBRAS OF DERIVATION TYPE

Notation 1. Let V, W real vector spaces, J_V a complex structure on V , J_W a complex structure on W . Then

$$\Lambda^{0,p} V^* \otimes W = \{ \lambda \in \Lambda^p V^* \otimes W : \lambda(v_1, \dots, J_V v_k, \dots, v_p) = -J_W \lambda(v_1, \dots, v_k, \dots, v_p) \}.$$

Definition 7. Let A be a \mathbb{R} -algebra with unit and \mathfrak{g} a A -module of derivations of A . A structure of complex A -Lie algebra of derivation type is a triple $(\mathfrak{g}, [\cdot, \cdot], J)$, where $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra verifying

$$[aV, W] = a[V, W] - (Wa)V$$

for every $a \in A$, $V, W \in \mathfrak{g}$ and J is a complex structure on \mathfrak{g} which is A -linear.

Definition 8. Let $(\mathfrak{g}, [\cdot, \cdot], J)$ be a complex A -Lie algebra of derivation type. We define the Nijenhuis tensor $N = N_{J, [\cdot, \cdot]}$ and $\bar{\partial} = \bar{\partial}_{J, [\cdot, \cdot]}$ by

$$N(V, W) = [JV, JW] - [V, W] - J[JV, W] - J[V, JW], \quad V, W \in \mathfrak{g}$$

and

$$(\bar{\partial}W)(V) = \frac{1}{2}([V, W] + J[JV, W]) + \frac{1}{4}N(V, W), \quad V, W \in \mathfrak{g}.$$

Remark 4. It is easy to see that N is A -bilinear and $N(JV, W) = N(V, JW) = -JN(V, W)$, so $N \in \Lambda^{0,2}\mathfrak{g}^* \otimes \mathfrak{g}$. Indeed

$$\begin{aligned} N(V, JW) &= [JV, -W] - [V, JW] - J[JV, JW] - J[V, -W] \\ &= -[V, JW] - [JV, W] + J[V, W] - J[JV, JW] = N(JV, W) \end{aligned}$$

and

$$N(JV, W) = -J([JV, JW] - [V, W] - J[JV, W] - J[V, JW]) = -JN(V, W).$$

Lemma 6. Let $(\mathfrak{g}, [\cdot, \cdot], J)$ be a complex A -Lie algebra of derivation type. Then:

- i) $\bar{\partial} : \Lambda^{0,0}\mathfrak{g}^* \otimes \mathfrak{g} \rightarrow \Lambda^{0,1}\mathfrak{g}^* \otimes \mathfrak{g}$;
- ii) $\bar{\partial}J - J\bar{\partial} = 0$;
- iii) $\bar{\partial}(aW)(V) = (\bar{\partial}a)(V)W + a(\bar{\partial}W)(V)$ where

$$(\bar{\partial}a)(V)W = \frac{1}{2}((Va)W + J(Va)JW).$$

Proof. i)

$$\begin{aligned} (\bar{\partial}W)(JV) &= \frac{1}{2}([JV, W] + J[-V, W]) + \frac{1}{4}N(JV, W) \\ &= -J\left(\frac{1}{2}(J[JV, W] + [V, W]) + \frac{1}{4}N(JV, W)\right) = -J(\bar{\partial}W)(V). \end{aligned}$$

ii) Since

$$N(V, JW) - J[V, W] + J[JV, JW] + [JV, W] + [V, JW] = 0,$$

we have

$$\begin{aligned} 2(\bar{\partial}JW)(V) &= [V, JW] + J[JV, JW] + \frac{1}{2}N(V, JW) \\ &= J[V, W] - [JV, W] - \frac{1}{2}N(V, JW) = J((\bar{\partial}W)(V)). \end{aligned}$$

iii)

$$\begin{aligned}
\bar{\partial}(aW)(V) &= \frac{1}{2}([V, aW] + J[JV, aW]) + \frac{1}{4}N(V, aW) \\
&= \frac{1}{2}(-a[W, V] + (Va)W + J(-a[W, JV] + (JVa)W)) + \frac{1}{4}aN(V, W) \\
&= \frac{1}{2}((Va)W + (JVa)JW) + a\left([V, W] + J[JV, W] + \frac{1}{4}N(V, W)\right) \\
&= (\bar{\partial}a)(V)W + a(\bar{\partial}W)(V).
\end{aligned}$$

□

Lemma 7. *Let V be a real vector space and J, \tilde{J} complex structures on J such that $\det(J + \tilde{J}) \neq 0$. There exists a unique $S \in \text{End}_{\mathbb{R}}(V)$ such that $\tilde{J} = (I + S)J(I + S)^{-1}$, $SJ + JS = 0$.*

Proof. Let $S \in \text{End}_{\mathbb{R}}(V)$ such that $\tilde{J} = (I + S)J(I + S)^{-1}$, $SJ + JS = 0$. Then $\tilde{J}(I + S) = (I + S)J$, so $\tilde{J} + \tilde{J}S = J + SJ = J - JS$ and it follows that $S = (J - \tilde{J})(J + \tilde{J})^{-1}$.

Conversely, define $S = (J - \tilde{J})(J + \tilde{J})^{-1}$. Since

$$\begin{aligned}
SJ + JS &= (J - \tilde{J})(J + \tilde{J})^{-1}J + J(J - \tilde{J})(J + \tilde{J})^{-1} \\
&= (J - \tilde{J})(I - \tilde{J}J)^{-1} + (-I - J\tilde{J})(J + \tilde{J})^{-1} \\
&= (J - \tilde{J})(I - \tilde{J}J)^{-1} + (\tilde{J} - J)\tilde{J}(J + \tilde{J})^{-1} \\
&= (J - \tilde{J})(I - \tilde{J}J)^{-1} + (\tilde{J} - J)(-\tilde{J}J + I)^{-1} = 0,
\end{aligned}$$

it follows that $\tilde{J} = (I + S)J(I + S)^{-1}$. □

Proposition 3. *Let $(\mathfrak{g}, [\cdot, \cdot], J)$, $(\mathfrak{g}, [\cdot, \cdot], \tilde{J})$ be complex A -Lie algebras of derivation type such that $\det(J + \tilde{J}) \neq 0$. Let $S \in \text{End}_{\mathbb{R}}(\mathfrak{g})$ such that $\tilde{J} = (I + S)J(I + S)^{-1}$, $SJ + JS = 0$. Then*

$$\begin{aligned}
N_{\tilde{J}}((I + S)V, (I + S)W) &= (I - S)^{-1}(N_J(V, W) + S(N_J(V, W) - N_J(SV, SW))) \\
(3.1) \quad &\quad -4(I - S)^{-1}\left(\bar{\partial}_J S + \frac{1}{2}[S, S]\right)(V, W)
\end{aligned}$$

where

$$\bar{\partial}_J S(V, W) = \bar{\partial}_J(SW)(V) - \bar{\partial}_J(SV)(W) - \frac{1}{2}S([V, W] - [JV, JW])$$

and

$$\begin{aligned}
[S, S](V, W) &= [SV, SW] - [JSV, JSW] - S([SV, W] + [V, SW] + J[V, JSW] + J[JSV, W]) \\
&\quad - \frac{1}{2}(SN_J(SV, W) + SN_J(V, SW) - N_J(SV, SW)).
\end{aligned}$$

Proof. Denote

$$\tilde{E}(V, W) = N_{\tilde{J}}((I + S)V, (I + S)W).$$

We have

$$\begin{aligned} \tilde{E}(V, W) &= [\tilde{J}(I + S)V, \tilde{J}(I + S)W] - [(I + S)V, (I + S)W] \\ &\quad - \tilde{J}[\tilde{J}(I + S)V, (I + S)W] - \tilde{J}[(I + S)V, \tilde{J}(I + S)W] \\ &= [(I + S)JV, (I + S)JW] - [(I + S)V, (I + S)W] \\ &\quad - (I + S)J(I + S)^{-1}[(I + S)JV, (I + S)W] \\ &\quad - (I + S)J(I + S)^{-1}[(I + S)V, (I + S)JW] \\ &= [JV, JW] + [SV, SW] + [JV, SJW] + [SV, JW] \\ &\quad - [V, W] - [SV, SW] - [V, SW] - [SV, W] \\ &\quad - (I + S)J(I + S)^{-1}([JV, W] + [SV, SW] + [JV, SW] + [SV, W]) \\ &\quad - (I + S)J(I + S)^{-1}([V, JW] + [SV, SJW] + [V, SJW] + [SV, JW]). \end{aligned}$$

But

$$\begin{aligned} (I + S)J(I + S)^{-1} &= (I - S)^{-1}((I - S)(I + S)J(I + S)^{-1}) \\ &= (I - S)^{-1}(I - S^2)J(I + S)^{-1} \\ &= (I - S)^{-1}(I + S)(I - S)J(I + S)^{-1} \\ &= (I - S)^{-1}(I + S)(J - SJ)(I + S)^{-1} \\ &= (I - S)^{-1}(I + S)(J + JS)(I + S)^{-1} \\ &= (I - S)^{-1}(I + S)J \end{aligned}$$

so

$$\begin{aligned} \tilde{E}(V, W) &= [JV, JW] + [JSV, JSW] - [JV, JSW] - [JSV, JW] \\ &\quad - [V, W] - [SV, SW] - [V, SW] - [SV, W] \\ &\quad - (I - S)^{-1}(I + S)J([JV, W] - [JSV, SW] + [JV, SW] - [JSV, W]) \\ (3.2) \quad &\quad - (I - S)^{-1}(I + S)J([V, JW] - [SV, JSW] - [V, JSW] + [SV, JW]) \end{aligned}$$

We set

$$E(V, W) = (I - S)\tilde{E}(V, W)$$

and from (3.2) we have

$$\begin{aligned} (3.3) \quad E(V, W) &= [JV, JW] + [JSV, JSW] - [JV, JSW] - [JSV, JW] \\ (3.4) \quad &\quad - S([JV, JW] + [JSV, JSW] - [JV, JSW] - [JSV, JW]) \\ (3.5) \quad &\quad - [V, W] - [SV, SW] - [V, SW] - [SV, W] \\ (3.6) \quad &\quad + S([V, W] + [SV, SW] + [V, SW] + [SV, W]) \\ (3.7) \quad &\quad - J([JV, W] - [JSV, SW] + [JV, SW] - [JSV, W]) \\ (3.8) \quad &\quad - SJ([JV, W] - [JSV, SW] + [JV, SW] - [JSV, W]) \\ (3.9) \quad &\quad - J([V, JW] - [SV, JSW] - [V, JSW] + [SV, JW]) \\ (3.10) \quad &\quad - SJ([V, JW] - [SV, JSW] - [V, JSW] + [SV, JW]). \end{aligned}$$

By adding the first terms, (respectively second terms, third terms and forth terms) in (3.3), (3.5), (3.7) and (3.9) and then in (3.4), (3.6), (3.8) and (3.10) we obtain the following form of $E(V, W)$:

$$\begin{aligned}
(3.11) \quad & [JV, JW] - [V, W] - J[JV, W] - J[V, JW] \\
(3.12) \quad & + [JSV, JSW] - [SV, SW] + J[JSV, SW] + J[SV, JSW] \\
(3.13) \quad & - [JV, JSW] - [V, SW] - J[JV, SW] + J[V, JSW] \\
(3.14) \quad & - [JSV, JW] - [SV, W] + J[JSV, W] - J[SV, JW] \\
(3.15) \quad & + S(-[JV, JW] + [V, W] - J[JV, W] - J[V, JW]) \\
(3.16) \quad & + S(-[JSV, JSW] + [SV, SW] + J[JSV, SW] + J[SV, JSW]) \\
(3.17) \quad & + S([JV, JSW] + [V, SW] - J[JV, SW] + J[V, JSW]) \\
(3.18) \quad & + S([JSV, JW] + [SV, W] + J[JSV, W] - J[SV, JW])
\end{aligned}$$

Now

$$\begin{aligned}
(3.11) \quad & = N_J(V, W) \\
(3.12) \quad & = N_J(SV, SW) + 2(J[JSV, SW] + J[SV, JSW]) \\
(3.13) \quad & = -N_J(V, SW) - 2([V, SW] + J[JV, SW]) \\
(3.14) \quad & = -N_J(SV, W) - 2([SV, W] + J[SV, JW]) \\
(3.15) \quad & = S(-[JV, JW] + [V, W] - J[JV, W] - J[V, JW]) \\
(3.16) \quad & = -SN_J(SV, SW) \\
(3.17) \quad & = SN_J(V, SW) + 2S([V, SW] + J[V, JSW]) \\
(3.18) \quad & = SN_J(SV, W) + 2S([SV, W] + J[JSV, W])
\end{aligned}$$

so

$$\begin{aligned}
(3.19) \quad E(V, W) \quad & = N_J(V, W) - SN_J(SV, SW) - 2[V, SW] \\
& - 2J[JV, SW] - 2[SV, W] - 2J[SV, JW] \\
& + S[V, W] - S[JV, JW] - N_J(V, SW) - N_J(SV, W) \\
& + 2S([V, SW] + J[V, JSW] + [SV, W] + J[JSV, W]) \\
& + SN_J(SV, W) + SN_J(V, SW) + N_J(SV, SW) \\
& + 2(J[JSV, SW] + J[SV, JSW]) + S(-J[JV, W] - J[V, JW])
\end{aligned}$$

Since

$$\begin{aligned}
\bar{\partial}_J S(V, W) \quad & = (\bar{\partial}_J SW)(V) - (\bar{\partial}_J SV)(W) - \frac{1}{2}S([V, W] - [JV, JW]) \\
& = \frac{1}{2}([V, SW] + J[JV, SW] - [W, SV] - J[JW, SV]) - \frac{1}{2}(S[V, W] - S[JV, JW]) \\
& + \frac{1}{4}(N_J(V, SW) - N_J(W, SV))
\end{aligned}$$

and

$$2(J[JSV, SW] + J[SV, JSW]) = -2N_J(SV, SW) + 2[JSV, JSW] - 2[SV, SW]$$

(3.19) becomes

$$\begin{aligned}
E(V, W) &= N_J(V, W) - SN_J(SV, SW) \\
&\quad -2[V, SW] - 2J[JV, SW] + 2[WS, V] + 2J[JW, SV] \\
&\quad +2(S[V, W] - S[JV, JW]) - (S[V, W] - S[JV, JW]) \\
&\quad -N_J(V, SW) + N_J(W, SV) \\
&\quad +2S([V, SW] + J[V, JSW] + [SV, W] + J[JSV, W]) \\
&\quad +SN_J(SV, W) + SN_J(V, SW) + N_J(SV, SW) \\
&\quad -2N_J(SV, SW) + 2[JSV, JSW] - 2[SV, SW] \\
&\quad +S(-J[JV, W] - J[V, JW]) \\
&= N_J(V, W) - SN_J(SV, SW) - 4\bar{\partial}_J S(V, W) - (S[V, W] - S[JV, JW]) \\
&\quad +2S([V, SW] + J[V, JSW] + [SV, W] + J[JSV, W]) \\
&\quad +2[JSV, JSW] - 2[SV, SW] + S(-J[JV, W] - J[V, JW]) \\
&\quad +SN_J(SV, W) + SN_J(V, SW) - N_J(SV, SW) \\
&= N_J(V, W) - SN_J(SV, SW) - 4\bar{\partial}_J S(V, W) \\
&\quad +2S([V, SW] + J[V, JSW] + [SV, W] + J[JSV, W]) \\
&\quad +S([JV, JW] - [V, W] - J[JV, W] - J[V, JW]) \\
&\quad +2[JSV, JSW] - 2[SV, SW] + SN_J(V, SW) + SN_J(SV, W) - N_J(SV, SW) \\
&= N_J(V, W) + SN_J(V, W) - SN_J(SV, SW) - 4\bar{\partial}_J S(V, W) \\
&\quad -2[SV, SW] + 2[JSV, JSW] + 2S([SV, W] + J[JSV, W] + [V, SW] + J[V, JSW]) \\
&\quad +SN_J(VS, W) + SN_J(V, SW) - N_J(SV, SW) \\
&= N_J(V, W) + SN_J(V, W) - SN_J(SV, SW) - 4\bar{\partial}_J S(V, W) - 2[S, S](V, W)
\end{aligned}$$

which is equivalent to (3.1). \square

From the Proposition 3 we obtain:

Corollary 3. *Let $(\mathfrak{g}, [\cdot, \cdot], J)$, $(\mathfrak{g}, [\cdot, \cdot], \tilde{J})$ be complex A -Lie algebras of derivation type such that $\det(J + \tilde{J}) \neq 0$. Let $S \in \text{End}_{\mathbb{R}}(\mathfrak{g})$ such that $\tilde{J} = (I + S)J(I + S)^{-1}$, $SJ + JS = 0$. Then $N_{\tilde{J}} = 0$ if and only if*

$$\bar{\partial}_J S + \frac{1}{2}[[S, S]] = \frac{1}{4}N_J$$

where $[[S, S]] = [S, S] - \frac{1}{2}S(N_J - N_J(S, S))$ and $N_J(S, S)(V, W) = N_J(SV, SW)$.

4. LEVI FLAT STRUCTURES

Definition 9. *Let L be a smooth manifold. A Levi flat structure on L is a couple (ξ, J) where $\xi \subset T(L)$ is an integrable distribution of codimension 1 and $J : \xi \rightarrow \xi$ defines a complex integrable structure on each leaf.*

Example 1. *By W. Lickorish [12] (and by an unpublished work of S. Novikov and H. Zieschang, 1965), any compact orientable 3-manifold has a foliation of codimension 1. By J. Wood [20] any compact 3-manifold has a transversally orientable foliation of codimension 1. It follows that that all compact orientable 3-manifolds admit a Levi flat structure.*

Notation 2. From now on we consider the following setting:

- L a smooth manifold;
- (ξ, J) a Levi flat structure on L ;
- (γ, X) a DGLA defining couple for ξ ;
- $\alpha \in \mathcal{Z}^1(L)$ such that $\xi_\alpha = \ker(\gamma + \alpha)$ is integrable;
- $\mathcal{H}(\xi)$ the algebra of vector fields on L which are tangent to ξ ;
- $[\cdot, \cdot]$ the Lie bracket;
- $\Lambda_J^{p,q}(\xi)$ the $p+q$ -forms α on L such that the restriction of α on each leaf \mathcal{L} of the Levi foliation endowed with the complex structure J is a (p, q) -form on \mathcal{L} and $\Lambda^k(\xi) = \bigoplus_{p+q=k} \Lambda_J^{p,q}(\xi)$.

Definition 10. For $V, W \in T(L)$, define $\omega_\alpha(V) = V - \alpha(V)X$ and

$$[V, W]_\alpha = \omega_\alpha^{-1}[\omega_\alpha(V), \omega_\alpha(W)].$$

Lemma 8. 1) $\omega_\alpha \in \text{End}T(L)$ and $\omega_\alpha(\xi) = \xi_\alpha$.

2) $((\mathcal{H}(\xi))_\alpha, [\cdot, \cdot]_\alpha, J)$ is a complex $C^\infty(L)$ -Lie algebra of derivation type, where $(\mathcal{H}(\xi))_\alpha$ is the $C^\infty(L)$ -algebra $\mathcal{H}(\xi)$ endowed with the derivation $\langle V, f \rangle_\alpha = \omega_\alpha(V)(f)$.

Proof. It is obvious that $\omega_\alpha^{-1}(V) = V + \alpha(V)X$, $(\gamma + \alpha)(V - \alpha(V)X) = 0$ for $V \in \xi$ and $\gamma(V + \alpha(V)X) = 0$ for $V \in \ker(\gamma + \alpha)$.

Let $V, W \in \mathcal{H}(\xi)$.

$$\begin{aligned} [\omega_\alpha(V), \omega_\alpha(W)] &= [V - \alpha(V)X, W - \alpha(W)X] \\ &= [V, W] - [\alpha(V)X, W] + [\alpha(W)X, V] + [\alpha(V)X, \alpha(W)X] \\ &= [V, W] - \alpha(V)[X, W] + W(\alpha(V))X + \alpha(W)[X, V] - V(\alpha(W))X \\ &\quad + [\alpha(V)X, \alpha(W)X] \end{aligned}$$

$$\begin{aligned} [\alpha(V)X, \alpha(W)X] &= \alpha(V)[X, \alpha(W)X] - (\alpha(W)X)(\alpha(V))X \\ &= -\alpha(V)[\alpha(W)X, X] - \alpha(W)(X(\alpha(V)))X \\ &= \alpha(V)X(\alpha(W))X - \alpha(W)(X(\alpha(V)))X \end{aligned}$$

So

$$\begin{aligned} [\omega_\alpha(V), \omega_\alpha(W)] &= [V, W] - \alpha(V)[X, W] + W(\alpha(V))X + \alpha(W)[X, V] - V(\alpha(W))X \\ &\quad + \alpha(V)X(\alpha(W))X - \alpha(W)(X(\alpha(V)))X \end{aligned}$$

$$\begin{aligned} [\omega_\alpha(aV), \omega_\alpha(W)] &= a[V, W] - W(a)V - a\alpha(V)[X, W] + W(a\alpha(V))X \\ &\quad - \alpha(W)[aV, X] - aV(\alpha(W))X + a\alpha(V)X(\alpha(W))X - \alpha(W)(X(a\alpha(V)))X \\ &= a[V, W] - W(a)V - a\alpha(V)[X, W] + W(a)\alpha(V)X + aW(\alpha(V))X \\ &\quad - a\alpha(W)[V, X] + \alpha(W)X(a)V - aV(\alpha(W))X + a\alpha(V)X(\alpha(W))X \\ &\quad - \alpha(W)(X(a)\alpha(V))X - a\alpha(W)(X(\alpha(V)))X \end{aligned}$$

$$\begin{aligned} [aV, W]_\alpha &= \omega_\alpha^{-1}[\omega_\alpha(aV), \omega_\alpha(W)] \\ &= [\omega_\alpha(aV), \omega_\alpha(W)] + \alpha([\omega_\alpha(aV), \omega_\alpha(W)])X \end{aligned}$$

Since

$$\begin{aligned} \alpha([\omega_\alpha(aV), \omega_\alpha(W)]) &= a\alpha([V, W]) - W(a)\alpha(V) - a\alpha(V)\alpha([X, W]) \\ &\quad - a\alpha(W)\alpha([V, X]) + \alpha(W)X(a)\alpha(V) \end{aligned}$$

we have

$$\begin{aligned}
[aV, W]_\alpha &= a[V, W] - W(a)V - a\alpha(V)[X, W] + W(a)\alpha(V)X + aW(\alpha(V))X \\
&\quad - a\alpha(W)[V, X] + \alpha(W)X(a)V - aV(\alpha(W))X + a\alpha(V)X(\alpha(W))X \\
&\quad - \alpha(W)(X(a)\alpha(V))X - a\alpha(W)(X(\alpha(V)))X \\
&\quad + a\alpha([V, W])X - W(a)\alpha(V)X - a\alpha(V)\alpha([X, W])X \\
&\quad - a\alpha(W)\alpha([V, X])X + \alpha(W)X(a)\alpha(V)X
\end{aligned}$$

For $a = 1$

$$\begin{aligned}
[V, W]_\alpha &= [V, W] - \alpha(V)[X, W] + W(\alpha(V))X \\
&\quad - \alpha(W)[V, X] - V(\alpha(W))X + \alpha(V)X(\alpha(W))X \\
&\quad - \alpha(W)(X(\alpha(V)))X \\
(4.1) \quad &\quad + \alpha([V, W])X - \alpha(V)\alpha([X, W])X \\
&\quad - \alpha(W)\alpha([V, X])X
\end{aligned}$$

So

$$\begin{aligned}
[aV, W]_\alpha &= a[V, W]_\alpha - W(a)V + W(a)\alpha(V)X \\
&\quad + \alpha(W)X(a)V \\
&\quad - \alpha(W)(X(a)\alpha(V))X \\
&\quad - W(a)\alpha(V)X \\
&\quad + \alpha(W)X(a)\alpha(V)X \\
&= a[V, W]_\alpha - W(a)V + \alpha(W)X(a)V \\
&= a[V, W]_\alpha - (W - \alpha(W)X)(a)V \\
&= a[V, W]_\alpha - \omega_\alpha(W)(a)V = a[V, W]_\alpha - \langle W, a \rangle_\alpha V.
\end{aligned}$$

□

5. THE $(0, 1)$ -FORM ASSOCIATED TO A DGLA DEFINING COUPLE

Notation 3. *i) $\bar{\partial} = \bar{\partial}_J : \Lambda_j^{0,p}(\xi) \otimes \xi \rightarrow \Lambda_j^{0,p+1}(\xi) \otimes \xi$ is the extension by linearity of*

$$\bar{\partial}(\alpha \otimes Z) = \bar{\partial}\alpha \otimes Z + (-1)^p \alpha \wedge \bar{\partial}Z;$$

ii) Let $\alpha \in \Lambda^1(\xi)$, $\beta \in \Lambda^1(\xi) \otimes \xi$ and $V \in \xi$. Then we denote $\alpha^{0,1} \in \Lambda_j^{0,1}(\xi)$ and $\beta^{0,1} \in \Lambda_j^{0,1}(\xi) \otimes \xi$ the forms defined by

$$\alpha^{0,1}(V) = \frac{1}{2}(\alpha(V) + i\alpha(JV)), \quad \beta^{0,1}(V) = \frac{1}{2}(\beta(V) + J\beta(JV));$$

iii) Let $\alpha \in \Lambda_j^{0,1}(\xi)$, $V, W \in \xi$. Then $\bar{\partial}\alpha \in \Lambda_j^{0,2}(\xi)$ is defined by $\bar{\partial}\alpha(V, W) = \frac{1}{4}d\alpha^{\mathbb{C}}(V + iJV, W + iJW)$ where $d\alpha^{\mathbb{C}} \in \Lambda_j^2\xi \otimes \mathbb{C}$ is the complexification of $d\alpha$.

Lemma 9. *Let $\omega \in \Lambda_j^{0,1}(\xi) \otimes \xi$, $V, W \in \xi$. Then*

$$\bar{\partial}\omega(V, W) = (\bar{\partial}(\omega(W))(V) - \bar{\partial}(\omega(V))(W)) - \frac{1}{2}\omega([V, W] - [JV, JW]).$$

Proof. It is enough to prove the Lemma for $\omega = \alpha^{0,1} \otimes Z$, where $\alpha \in \Lambda_j^1(\xi)$ and $Z \in \xi$.

Then

$$\begin{aligned}
 \bar{\partial}(\alpha^{0,1} \otimes Z)(V, W) &= (\bar{\partial}\alpha^{0,1} \otimes Z - \alpha^{0,1} \wedge \bar{\partial}Z)(V, W) \\
 (5.1) \quad &= \frac{1}{4}d\alpha^{\mathbb{C}}(V + iJV, W + iJW)Z - \alpha^{0,1}(V)(\bar{\partial}Z)(W) + \alpha^{0,1}(W)(\bar{\partial}Z)(V)
 \end{aligned}$$

But

$$\begin{aligned}
 d\alpha^{\mathbb{C}}(V + iJV, W + iJW) &= (V + iJV)(\alpha^{\mathbb{C}}(W + iJW)) - (W + iJW)(\alpha^{\mathbb{C}}(V + iJV)) - \alpha^{\mathbb{C}}[V + iJV, W + iJW] \\
 &= (V + iJV)(\alpha(W) + i\alpha(JW)) - (W + iJW)(\alpha(V) + i\alpha(JV)) \\
 &\quad - \alpha^{\mathbb{C}}([V, W] - [JV, JW] + i[JV, W] + [V, JW]) \\
 &= V(\alpha(W)) - (JV)\alpha(JW) + i((JV)(\alpha(W)) + V(\alpha(JW))) \\
 &\quad - (W(\alpha(V)) - (JW)\alpha(JV) + i((JW)(\alpha(V)) + W(\alpha(JV)))) \\
 &\quad - \alpha([V, W] - [JV, JW]) + i\alpha([JV, W] + [V, JW])
 \end{aligned}$$

and so

$$\begin{aligned}
 d\alpha^{\mathbb{C}}(V + iJV, W + iJW)Z &= V(\alpha(W)) - (JV)\alpha(JW)Z + (JV)(\alpha(W)) + V(\alpha(JW))JZ \\
 (5.2) \quad &\quad - ((W(\alpha(V)) - (JW)\alpha(JV))Z + (JW)(\alpha(V)) + W(\alpha(JV))JZ) \\
 &\quad - (\alpha([V, W] - [JV, JW]))Z + \alpha([JV, W] + [V, JW])JZ.
 \end{aligned}$$

We have also

$$\begin{aligned}
 \alpha^{0,1}(V)(\bar{\partial}Z)(W) &= \frac{1}{4}(\alpha(V) + i\alpha(JV))([W, Z] + J[JW, Z]) \\
 (5.3) \quad &= \frac{1}{4}(\alpha(V)[W, Z] + \alpha(V)J[JW, Z] + \alpha(JV)J[W, Z] - \alpha(JV)[JW, Z]),
 \end{aligned}$$

$$\begin{aligned}
 \alpha^{0,1}(W)(\bar{\partial}Z)(V) &= \frac{1}{4}(\alpha(W) + i\alpha(JW))([V, Z] + J[JV, Z]) \\
 (5.4) \quad &= \frac{1}{4}(\alpha(W)[V, Z] + \alpha(W)J[JV, Z] + \alpha(JW)J[V, Z] - \alpha(JW)[JV, Z])
 \end{aligned}$$

so from (5.1), (5.2), (5.3) and (5.4) it follows that

$$\begin{aligned}
 \bar{\partial}(\alpha^{0,1} \otimes Z)(V, W) &= \frac{1}{4}(V(\alpha(W)) - (JV)\alpha(JW)Z + (JV)(\alpha(W)) + V(\alpha(JW))JZ) \\
 &\quad - \frac{1}{4}((W(\alpha(V)) - (JW)\alpha(JV))Z + (JW)(\alpha(V)) + W(\alpha(JV))JZ) \\
 &\quad - \frac{1}{4}(\alpha([V, W] - [JV, JW]))Z + \alpha([JV, W] + [V, JW])JZ \\
 (5.5) \quad &\quad - \frac{1}{4}(\alpha(V) + i\alpha(JV))([W, Z] + J[JW, Z]) \\
 &\quad + \frac{1}{4}(\alpha(W) + i\alpha(JW))([V, Z] + J[JV, Z]).
 \end{aligned}$$

Since

$$\begin{aligned}
\bar{\partial}((\alpha^{0,1} \otimes Z)(W))(V) &= \bar{\partial}(\alpha^{0,1}(W)Z)(V) = \frac{1}{2}([V, \alpha^{0,1}(W)Z] + J[JV, \alpha^{0,1}(W)Z]) \\
&= \frac{1}{4}([V, (\alpha(W) + i\alpha(JW))Z] + J[JV, (\alpha(W) + i\alpha(JW))Z]) \\
&= \frac{1}{4}([V, \alpha(W)Z + \alpha(JW)JZ] + J[JV, \alpha(W)Z + \alpha(JW)JZ]) \\
&= \frac{1}{4}(\alpha(W)[V, Z] + \alpha(JW)[V, JZ] + \alpha(W)J[JV, Z] + \alpha(JW)J[JV, JZ]) \\
(5.6) \quad &+ \frac{1}{4}(V(\alpha(W)))Z + V(\alpha(JW)JZ + (JV)(\alpha(W))JZ - (JV)(\alpha(JW))Z)
\end{aligned}$$

and similarly

$$\begin{aligned}
\bar{\partial}((\alpha^{0,1} \otimes Z)(W))(V) &= \frac{1}{4}(\alpha(V)[W, Z] + \alpha(JV)[W, JZ] + \alpha(V)J[JW, Z] + \alpha(JV)J[JW, JZ]) \\
(5.7) \quad &+ \frac{1}{4}(W(\alpha(V)))Z + W(\alpha(JV)JZ + (JW)(\alpha(V))JZ - (JW)(\alpha(JV))Z)
\end{aligned}$$

from (5.5), (5.6) and (5.7) we obtain

$$\begin{aligned}
&4(\bar{\partial}(\alpha^{0,1} \otimes Z)(V, W) - 4(\bar{\partial}((\alpha^{0,1} \otimes Z)(W))(V) - \bar{\partial}((\alpha^{0,1} \otimes Z)(V))(W))) \\
&= V(\alpha(W)) - (JV)(\alpha(JW))Z + (JV)(\alpha(W)) + V(\alpha(JW))JZ \\
&\quad - \left(\left(W(\alpha(V)) - (JW)\alpha(JV) \right) Z + (JW)(\alpha(V)) + W(\alpha(JV))JZ \right) \\
&\quad - (\alpha([V, W] - [JV, JW]))Z + \alpha([JV, W] + [V, JW])JZ \\
&\quad - \left(\alpha(V)[W, Z] + \alpha(V)J[JW, Z] + \alpha(JV)J[W, Z] - \alpha(JV)[JW, Z] \right) \\
&\quad + \alpha(W)[V, Z] + \alpha(W)J[JV, Z] + \alpha(JW)J[V, Z] - \alpha(JW)[JV, Z] \\
&\quad - \left(\alpha(W)[V, Z] + \alpha(JW)[V, JZ] + \alpha(W)J[JV, Z] + \alpha(JW)J[JV, JZ] \right) \\
&\quad - (V(\alpha(W)))Z + V(\alpha(JW))JZ + (JV)(\alpha(W))JZ - (JV)(\alpha(JW))Z \\
&\quad + \alpha(V)[W, Z] + \alpha(JV)[W, JZ] + \alpha(V)J[JW, Z] + \alpha(JV)J[JW, JZ] \\
&\quad + (W(\alpha(V)))Z + W(\alpha(JV))JZ + (JW)(\alpha(V))JZ - (JW)(\alpha(JV))Z.
\end{aligned}$$

By reducing the terms having the same index, this last formula becomes

$$\begin{aligned}
&4(\bar{\partial}(\alpha^{0,1} \otimes Z)(V, W) - 4(\bar{\partial}((\alpha^{0,1} \otimes Z)(W))(V) - \bar{\partial}((\alpha^{0,1} \otimes Z)(V))(W))) \\
&= -(\alpha([V, W] - [JV, JW]))Z + \alpha([JV, W] + [V, JW])JZ \\
&\quad + \alpha(JV)(-J[W, Z] + [JW, Z] + [W, JZ] + J[JW, JZ]) \\
&\quad + \alpha(JW)(J[V, Z] - [JV, Z] - [V, JZ] - J[JV, JZ]) \\
(5.8) \quad &
\end{aligned}$$

Since $N_J(V, Z) = N_J(W, Z) = 0$, we have

$$\begin{aligned}
J[JW, JZ] - J[W, Z] + [JW, Z] + [W, JZ] &= 0 \\
J[JV, JZ] - J[V, Z] + [V, JZ] + [JV, Z] &= 0
\end{aligned}$$

and from (5.8) it follows that.

$$\begin{aligned} & 4 \left(\bar{\partial} (\alpha^{0,1} \otimes Z) (V, W) - 4 \left(\bar{\partial} ((\alpha^{0,1} \otimes Z) (W)) (V) - \bar{\partial} ((\alpha^{0,1} \otimes Z) (V)) (W) \right) \right) \\ &= -(\alpha([V, W] - [JV, JW])) Z + \alpha([JV, W] + [V, JW]) JZ. \end{aligned} \quad (5.9)$$

But

$$\begin{aligned} 2 (\alpha^{0,1} \otimes Z) ([V, W] - [JV, JW]) &= (\alpha([V, W] - [JV, JW])) + i\alpha(J([V, W] - [JV, JW])) Z \\ &= ((\alpha([V, W] - [JV, JW]) Z + \alpha(J([V, W] - [JV, JW]))) JZ \end{aligned}$$

and by (5.9) the Lemma follows. \square

Definition 11. Let Y a vector field on L . Define $T_Y = T_{Y, \gamma, X} \in \text{Hom}_{\mathbb{C}}(\xi, \xi)$ and $H_Y = H_{Y, J, \gamma, X} \in \Lambda_J^{0,1}(\xi) \otimes \xi$ by

$$(5.10) \quad T_Y(V) = [V, Y] - \gamma([V, Y]) X,$$

$$(5.11) \quad H_Y(V) = \frac{1}{2} (T_Y(V) + JT_Y(JV)) = \frac{1}{2} ([V, Y] - \gamma([V, Y]) X + J([JV, Y] - \gamma([JV, Y]) X)).$$

Remark 5. $i) H_Y$ is $C^\infty(L)$ -linear. Indeed, let $a \in C^\infty(L)$. Then

$$T_Y(aV) = [aV, Y] - \gamma([aV, Y]) X = a[V, Y] - Y(a)V - \gamma(a[V, Y] - Y(a)V) X = aT_Y(V) - Y(a)V$$

and

$$H_Y(aV) = \frac{1}{2} (T_Y(aV) + JT_Y(JaV)) = \frac{1}{2} (aT_Y(V) - Y(a)V + aJT_Y(JV) - J(Y(a)JV)) = aH_Y(V).$$

$$ii) H_V = \bar{\partial} V \text{ for } V \in \xi.$$

Definition 12. In the particular case $Y = X$ of Definition 11, we will note $T = T_X = T_{\gamma, X}$ and $H = H_X = H_{J, \gamma, X}$. $T_{\gamma, X}$ will be called the endomorphism associated to the DGLA defining couple (γ, X) and $H_{J, \gamma, X}$ will be called the $(0, 1)$ -form associated to the DGLA defining couple (γ, X) .

Remark 6. Let $V \in \xi$. We have

$$(5.12) \quad H(V) = \frac{1}{2} ([V, X] - \gamma([V, X]) X + J([JV, X] - \gamma([JV, X]) X)).$$

Since

$$(5.13) \quad \iota_X d\gamma(V) = d\gamma(X, V) = X(\gamma(V)) - V(\gamma(X)) - \gamma([X, V]) = \gamma([V, X])$$

it follows that

$$(5.14) \quad H(V) = \frac{1}{2} ([V, X] + J[JV, X]) + (\iota_X d\gamma)^{0,1}(V) X$$

Lemma 10.

$$\bar{\partial} H = (\iota_X d\gamma)^{0,1} \wedge H$$

where H is the $(0, 1)$ -form associated to the DGLA defining couple (γ, X) .

Proof. Let $V, W \in \xi$. By Lemma 9 we have

$$\begin{aligned} 2\bar{\partial} H(V, W) &= 2(\bar{\partial}(HW)(V) - \bar{\partial}(HV)(W)) - H([V, W] - [JV, JW]) \\ &= [V, HW] + J[JV, HW] - ([W, HV] + J[JW, HV]) - H([V, W] - [JV, JW]). \end{aligned}$$

By replacing HV , HW , $H([V, W])$ and $H[JV, JW]$ from the definition of H we obtain that $2\bar{\partial}H(V, W)$ is the sum of the following 12 terms:

$$\begin{aligned}
4\bar{\partial}H(V, W) = & \underset{(1)}{[V, [W, X] - \gamma([W, X])X]} + \underset{(2)}{[V, J([JW, X] - \gamma([JW, X])X)]} \\
& + \underset{(3)}{J([JV, [W, X] - \gamma([W, X])X)]} + \underset{(4)}{J[JV, J([JW, X] - \gamma([JW, X])X)]} \\
& - \underset{(5)}{[W, [V, X] - \gamma([V, X])X]} - \underset{(6)}{[W, J([JV, X] - \gamma([JV, X])X)]} \\
& - \underset{(7)}{J([JW, [V, X] - \gamma([V, X])X)]} - \underset{(8)}{J[JW, J([JV, X] - \gamma([JV, X])X)]} \\
& - \underset{(9)}{[[V, W], X] - \gamma([V, W], X)X} - \underset{(10)}{J([J[V, W], X] - \gamma([J[V, W], X])X)} \\
& + \underset{(11)}{[JV, JW], X] - \gamma([JV, JW], X)X} + \underset{(12)}{J([J[JV, JW], X] - \gamma([J[JV, JW], X])X)} \\
\end{aligned} \tag{5.15}$$

By using that $N(V, [JW, X] - \gamma([JW, X])X) = 0$, $N(W, J([JV, X] - \gamma([JV, X])X)) = 0$ and respectively $N(V, W) = 0$ we have

$$\begin{aligned}
(5.16) \quad & [V, J([JW, X] - \gamma([JW, X])X)] + J[JV, J([JW, X] - \gamma([JW, X])X)] \\
= & J[V, [JW, X] - \gamma([JW, X])X] - [JV, [JW, X] - \gamma([JW, X])X],
\end{aligned}$$

$$\begin{aligned}
(5.17) \quad & -J[JW, J([JV, X] - \gamma([JV, X])X)] - [W, J([JV, X] - \gamma([JV, X])X)] \\
= & [JW, [JV, X] - \gamma([JV, X])X] - J[W, [JV, X] - \gamma([JV, X])X],
\end{aligned}$$

and

$$(5.18) \quad [J[JV, JW], X] - [J[V, W], X] = -[[JV, W], X] - [[V, JW], X],$$

From (5.18) we have also

$$\begin{aligned}
& J[J[JV, JW], X] - \gamma([J[JV, JW], X])X - J([J[V, W], X] - \gamma([J[V, W], X])X) \\
= & -J([JV, W], X) - \gamma([JV, W], X)X - J([V, JW], X) - \gamma([V, JW], X)X. \\
\end{aligned} \tag{5.19}$$

Replacing now in (5.15) the terms (2) + (4) by (5.16), the terms (6) + (8) by (5.17) and the terms (10) + (12) by (5.18) and (5.19) we deduce

$$\begin{aligned}
(5.20) \quad 4\bar{\partial}H(V, W) &= [V, [W, X] - \underset{(\alpha)}{\gamma}([W, X])X] \\
&\quad + J[V, [JW, X] - \underset{(\beta)}{\gamma}([JW, X])X] \\
&\quad J[JV, [W, X] - \underset{(\gamma)}{\gamma}([W, X])X] \\
&\quad - [JV, [JW, X] - \underset{(\delta)}{\gamma}([JW, X])X] \\
&\quad - [W, [V, X] - \underset{(\alpha)}{\gamma}([V, X])X] \\
&\quad + [JW, [JV, X] - \underset{(\delta)}{\gamma}([JV, X])X] \\
&\quad - J[JW, [V, X] - \underset{(\beta)}{\gamma}([V, X])X] \\
&\quad - J[W, [JV, X] - \underset{(\gamma)}{\gamma}([JV, X])X] \\
&\quad - [[V, W], X] + \underset{(\alpha)}{\gamma}([[V, W], X])X \\
&\quad + [[JV, JW], X] - \underset{(\delta)}{\gamma}([[JV, JW], X])X \\
&\quad - J([[JV, W], X] - \underset{(\gamma)}{\gamma}([[JV, W], X])X) \\
&\quad - J([[V, JW], X] - \underset{(\beta)}{\gamma}([[V, JW], X])X).
\end{aligned}$$

By applying the Jacobi identities (5.21), (5.22), (5.23), (5.24) below

$$\begin{aligned}
&[V, [W, X] - \gamma([W, X])X] - [W, [V, X] - \gamma([V, X])X] - [[V, W], X] + \gamma([[V, W], X])X \\
&= -[V, \gamma([W, X])X] + [W, \gamma([V, X])X] + \gamma([[V, W], X])X \\
&\quad (5.21)
\end{aligned}$$

$$\begin{aligned}
&[V, [JW, X] - \gamma([JW, X])X] - [JW, [V, X] - \gamma([V, X])X] - [[V, JW], X] \\
&= -[V, \gamma([JW, X])X] + [JW, \gamma([V, X])X] - \gamma([[V, JW], X])X \\
&\quad (5.22)
\end{aligned}$$

$$\begin{aligned}
&[JV, [W, X] - \gamma([W, X])X] - [W, [JV, X] - \gamma([JV, X])X] - ([[JV, W], X] - \gamma([[JV, W], X])X) \\
&= -[JV, \gamma([W, X])X] + [W, \gamma([JV, X])X] + \gamma([[JV, W], X])X \\
&\quad (5.23)
\end{aligned}$$

$$\begin{aligned}
&-[JV, [JW, X] - \gamma([JW, X])X] + [JW, [JV, X] - \gamma([JV, X])X] + [[JV, JW], X] - \gamma([[JV, JW], X])X \\
&= [JV, \gamma([JW, X])X] - [JW, \gamma([JV, X])X] - \gamma([[JV, JW], X])X \\
&\quad (5.24)
\end{aligned}$$

for the pairs of 3 terms denoted (α) , (β) , (γ) and (δ) in (5.20) we obtain

$$\begin{aligned} 4\bar{\partial}H(V, W) = & -[V, \gamma([W, X])X] + [W, \gamma([V, X])X] + \gamma([V, W], X)X \\ & + J(-[V, \gamma([JW, X])X] + [JW, \gamma([V, X])X] - \gamma([V, JW], X)X) \\ & + J(-[JV, \gamma([W, X])X] + [W, \gamma([JV, X])X] + \gamma([JV, W], X)X) \\ & + [JV, \gamma([JW, X])X] - [JW, \gamma([JV, X])X] - \gamma([JV, JW], X)X. \end{aligned}$$

So

$$4\bar{\partial}H(V, W) = A + B + C + D$$

where

$$\begin{aligned} A &= -\gamma([W, X])[V, X] - V(\gamma([W, X]))X \\ &\quad + \gamma([V, X])[W, X] + W(\gamma([V, X]))X + \gamma([V, W], X)X \\ B &= J\left(-\gamma([JW, X])[V, X] - V(\gamma([JW, X]))X\right) \\ &\quad + J\left(\gamma([V, X])[JW, X] + JW(\gamma([V, X]))X - \gamma([V, JW], X)X\right) \\ C &= J\left(-\gamma([W, X])[JV, X] - (JV)(\gamma([W, X]))X\right) \\ &\quad + J\left(\gamma([JV, X])[W, X] + W(\gamma([JV, X]))X + \gamma([JV, W], X)X\right) \\ D &= \gamma([JW, X])[JV, X] + (JV)(\gamma([JW, X]))X \\ &\quad - \gamma([JV, X])[JW, X] - (JW)(\gamma([JV, X]))X - \gamma([JV, JW], X)X. \end{aligned}$$

By Lemma 5 $\iota_X d\gamma$ is d_b -closed, therefore by (2.12)

$$d_b \iota_X d\gamma = d\iota_X d\gamma - \gamma \wedge \iota_X d(\iota_X d\gamma) = 0.$$

It follows that

$$(5.25) \quad d\iota_X d\gamma(V, W) = 0.$$

By using (5.13) we have

$$\begin{aligned} d(\iota_X d\gamma)(V, W) &= V((\iota_X d\gamma)(W)) - W((\iota_X d\gamma)(V)) - (\iota_X d\gamma)[V, W] \\ &= V(d\gamma(X, W)) - W(d\gamma(X, V)) - d\gamma(X, [V, W]) \\ &= V(\gamma[W, X]) - W(\gamma[V, X]) - \gamma([V, W], X). \end{aligned}$$

So from (5.25) it follows

$$(5.26) \quad -V(\gamma[W, X]) + W(\gamma[V, X]) + \gamma([V, W], X) = 0$$

and similarly

$$(5.27) \quad -(JV)(\gamma[W, X]) + W(\gamma[JV, X]) + \gamma([JV, W], X) = 0,$$

$$(5.28) \quad -V(\gamma[JW, X]) + (JW)(\gamma[V, X]) + \gamma([V, JW], X) = 0,$$

$$(5.29) \quad - (JV) (\gamma [JW, X]) + (JW) (\gamma [JV, X]) + \gamma ([JV, JW], X) = 0,$$

By using (5.26), (5.27), (5.28) and (5.29) the pairs of terms denoted by (φ) , (ψ) , (η) and (ω) reduce in A, B, C, D respectively and the expression of $4\bar{\partial}H(V, W)$ becomes

$$(5.30) \quad \begin{aligned} 4\bar{\partial}H(V, W) &= -\gamma([W, X])[V, X] + \gamma([V, X])[W, X] \\ &\quad + J(-\gamma([JW, X])[V, X] + \gamma([V, X])[JW, X]) \\ &\quad + J(-\gamma([W, X])[JV, X] + \gamma([JV, X])[W, X]) \\ &\quad + \gamma([JW, X])[JV, X] - \gamma([JV, X])[JW, X] \end{aligned}$$

We compute now

$$\begin{aligned} &4 \left((\iota_X d\gamma)^{0,1} \wedge H \right) (V, W) \\ &= 4 \left((\iota_X d\gamma)^{0,1} (V) HW - (\iota_X d\gamma)^{0,1} (W) HV \right) \\ &= 2 \left((\iota_X d\gamma (V) + i\iota_X d\gamma (JV)) HW - (\iota_X d\gamma (W) + i\iota_X d\gamma (JW)) HV \right) \\ &= 2 (\iota_X d\gamma (V) HW + \iota_X d\gamma (JV) JHW - (\iota_X d\gamma (W) HV + \iota_X d\gamma (JW) JHV). \end{aligned}$$

By using (5.12) and (5.13) we obtain

$$\begin{aligned} &4 \left((\iota_X d\gamma)^{0,1} \wedge H \right) (V, W) \\ &= \gamma([V, X])([W, X] - \gamma([W, X])X + J([JW, X] - \gamma([JW, X])X)) \\ &\quad + \gamma([JV, X])J([W, X] - \gamma([W, X])X - ([JW, X] - \gamma([JW, X])X)) \\ &\quad - \gamma([W, X])([V, X] - \gamma([V, X])X + J([JV, X] - \gamma([JV, X])X)) \\ &\quad - \gamma([JW, X])(J([V, X] - \gamma([V, X])X - ([JV, X] - \gamma([JV, X])X)) \\ &= \gamma([V, X])[W, X] - \gamma([V, X])\underset{(1)}{\gamma([W, X])X} \\ &\quad - \gamma([JV, X])[JW, X] + \gamma([JV, X])\underset{(2)}{\gamma([JW, X])X} \\ &\quad - \gamma([W, X])[V, X] + \gamma([W, X])\underset{(1)}{\gamma([V, X])X} \\ &\quad + \gamma([JW, X])[JV, X] - \gamma([JW, X])\underset{(2)}{\gamma([JV, X])X} \\ &\quad + J \left(\gamma([V, X])[JW, X] - \gamma([V, X])\underset{(3)}{\gamma([JW, X])X} \right) \\ &\quad + J \left(\gamma([JV, X])[W, X] - \gamma([JV, X])\underset{(4)}{\gamma([W, X])X} \right) \\ &\quad - J \left(\gamma([W, X])[JV, X] - \gamma([W, X])\underset{(4)}{\gamma([JV, X])X} \right) \\ &\quad - J \left(\gamma([JW, X])[V, X] - \gamma([JW, X])\underset{(3)}{\gamma([V, X])X} \right). \end{aligned}$$

After reducing the pairs of terms (1), (2), (3), (4), this expression coincides with (5.30) and the Lemma is proved. \square

Proposition 4. *Let $(\gamma, X), (\hat{\gamma}, \hat{X})$ be DGLA-defining couples, $\hat{\gamma} = e^\lambda \gamma$, $\hat{X} = e^{-\lambda} X + U$, $\lambda \in C^\infty(L)$, $U \in \xi$. Then:*

$$H_{J, \hat{\gamma}, \hat{X}} = e^{-\lambda} H_{J, \gamma, X} + \bar{\partial} U - \left((\iota_X d\gamma)^{0,1} - \bar{\partial} \lambda \right) \otimes U.$$

Proof. Let $V \in \xi$. We have

$$\begin{aligned} T_{\hat{\gamma}, \hat{X}}(V) &= [V, e^{-\lambda} X + U] - e^\lambda \gamma([V, e^{-\lambda} X + U])(e^{-\lambda} X + U) \\ &= e^{-\lambda} [V, X] + V(e^{-\lambda} X) - \gamma(e^{-\lambda} [V, X] + V(e^{-\lambda} X)) X \\ &\quad + [V, U] - \gamma([V, U]) X - e^\lambda \gamma(e^{-\lambda} [V, X] + V(e^{-\lambda} X) + [V, U]) U. \end{aligned}$$

Since $[V, U] \in \xi$ and $\gamma(X) = 1$ we obtain

$$T_{\hat{\gamma}, \hat{X}}(V) = e^{-\lambda} T_{\gamma, X}(V) + [V, U] - (\gamma[V, X] + e^\lambda V(e^{-\lambda})) U.$$

It follows that

$$\begin{aligned} H_{J, \hat{\gamma}, \hat{X}}(V) &= \frac{1}{2} (T_{\hat{\gamma}, \hat{X}}(V) + J T_{\hat{\gamma}, \hat{X}}(JV)) \\ &= e^{-\lambda} H_{J, \gamma, X}(V) + \frac{1}{2} ([V, U] + J([JV, U])) - \frac{1}{2} (\gamma[V, X] U + \gamma([JV, X]) JU) \\ &\quad - \frac{1}{2} e^\lambda V(e^{-\lambda}) U - \frac{1}{2} e^\lambda JV(e^{-\lambda}) JU \\ &= e^{-\lambda} H_{J, \gamma, X}(V) + \bar{\partial} U(V) - \frac{1}{2} (\gamma[V, X] U + \gamma([JV, X]) JU) \\ &\quad - e^\lambda ((d(e^{-\lambda})(V)) U + (d(e^{-\lambda})(JV)) JU). \end{aligned}$$

By (5.13) we obtain

$$\begin{aligned} H_{J, \hat{\gamma}, \hat{X}}(V) &= e^{-\lambda} H_{J, \gamma, X}(V) + \bar{\partial} U(V) - \frac{1}{2} (\iota_X d\gamma(V) U + \iota_X d\gamma(JV) JU) \\ &\quad + \frac{1}{2} ((d\lambda(V)) U + (d\lambda(JV)) JU) \\ &= e^{-\lambda} H_{J, \gamma, X}(V) + \bar{\partial} U(V) - \left((\iota_X d\gamma - d\lambda)^{0,1}(V) \right) U \\ &= \left(e^{-\lambda} H_{J, \gamma, X} + \bar{\partial} U + \left((\iota_X d\gamma)^{0,1} + \bar{\partial} \lambda \right) \otimes U \right) (V) \end{aligned}$$

and the Proposition is proved. \square

6. THE Ξ -COMPLEX

Remark 7. $(\iota_X d\gamma)^{0,1}$ is $\bar{\partial}_J$ -closed. Indeed, by Lemma 5,

$$\bar{\partial}_J (\iota_X d\gamma)^{0,1} = (d_b \iota_X d\gamma)^{0,2} = 0.$$

Definition 13. Let

$$\Xi = \Xi_{J, \gamma, X} : \Lambda^{0,*}(\xi) \otimes \xi \rightarrow \Lambda^{0,*}(\xi) \otimes \xi$$

defined by

$$\Xi P = \bar{\partial}_J P - (\iota_X d\gamma)^{0,1} \wedge P, \quad P \in \Lambda^{0,p}(\xi) \otimes \xi$$

Proposition 5. a) $\Xi^2 = 0$;

b) $\Xi H = 0$.

Proof. a) By Remark 7 we have

$$\begin{aligned}\bar{\Xi}^2 P &= \bar{\partial}_J \bar{\Xi} P - (\iota_X d\gamma)^{0,1} \wedge \bar{\Xi} P \\ &= \bar{\partial}_J \left(\bar{\partial}_J P - (\iota_X d\gamma)^{0,1} \wedge P \right) - (\iota_X d\gamma)^{0,1} \wedge \left(\bar{\partial}_J P - (\iota_X d\gamma)^{0,1} \wedge P \right) \\ &= (\iota_X d\gamma)^{0,1} \wedge \bar{\partial}_J P - (\iota_X d\gamma)^{0,1} \wedge \bar{\partial}_J P = 0\end{aligned}$$

b) follows by Lemma 10. \square

Proposition 6. *Let $(\gamma, X), (\hat{\gamma}, \hat{X})$ be DGLA-defining couples, $\hat{\gamma} = e^\lambda \gamma$, $\hat{X} = e^{-\lambda} X + U$, $\lambda \in C^\infty(L)$, $U \in \xi$. Then:*

$$\bar{\Xi}_{J, \hat{\gamma}, \hat{X}} e^{-\lambda} P = e^{-\lambda} \bar{\Xi}_{J, \gamma, X} P, \quad P \in \Lambda^{0,*}(\xi) \otimes \xi.$$

In particular the map $\Phi : \alpha \mapsto e^{-\lambda} \alpha$ induces an isomorphism $\hat{\Phi} : H^{0,}(\Lambda^{0,*}(\xi) \otimes \xi, \bar{\Xi}_{J, \gamma, X}) \rightarrow H^{0,*}(\Lambda^{0,*}(\xi) \otimes \xi, \bar{\Xi}_{J, \hat{\gamma}, \hat{X}})$.*

Proof. Let $V \in \xi$. Since $\xi = \ker \gamma$ and $V, U, [V, U] \in \xi$ and $\gamma(X) = 1$ we have

$$\begin{aligned}\iota_{\hat{X}} d\hat{\gamma}(V) &= d(e^\lambda \gamma)(e^{-\lambda} X + U, V) = e^\lambda d\gamma(e^{-\lambda} X + U, V) + e^\lambda d\lambda \wedge \gamma(e^{-\lambda} X + U, V) \\ &= e^\lambda ((-V(\gamma(e^{-\lambda} X + U))) - \gamma[e^{-\lambda} X + U, V]) - d\lambda(V) \gamma(e^{-\lambda} X + U) \\ &= e^\lambda ((-V(e^{-\lambda}) - e^{-\lambda} \gamma[X, V]) + V(e^{-\lambda}) - e^{-\lambda} d\lambda(V)) \\ &= -\gamma[X, V] - d\lambda(V) = \iota_X d\gamma(V) - d\lambda(V).\end{aligned}$$

So

$$\begin{aligned}\bar{\Xi}_{J, \hat{\gamma}, \hat{X}} P &= \bar{\partial}_J P - (\iota_{\hat{X}} d\hat{\gamma})^{0,1} \wedge P = \bar{\partial}_J P - (\iota_X d\gamma - d\lambda)^{0,1} \wedge P \\ &= \bar{\Xi}_{J, \gamma, X} P + \bar{\partial}\lambda \wedge P\end{aligned}$$

and

$$\begin{aligned}\bar{\Xi}_{J, \hat{\gamma}, \hat{X}}(e^{-\lambda} P) &= \bar{\Xi}_{J, \gamma, X}(e^{-\lambda} P) + \bar{\partial}\lambda \wedge e^{-\lambda} P = \bar{\partial}_J(e^{-\lambda} P) - (\iota_X d\gamma)^{0,1} \wedge e^{-\lambda} P + \bar{\partial}\lambda \wedge e^{-\lambda} P \\ &= e^{-\lambda} \bar{\partial}_J P - e^{-\lambda} \bar{\partial}\lambda \wedge e^{-\lambda} P - e^{-\lambda} (\iota_X d\gamma)^{0,1} \wedge P + e^{-\lambda} \bar{\partial}\lambda \wedge P = e^{-\lambda} \bar{\Xi}_{J, \gamma, X} P.\end{aligned}$$

\square

7. EXACT LEVI FLAT STRUCTURES

Lemma 11. *Let (ξ, J) a Levi flat structure. We denote $[H_{J, \gamma, X}] \in H^{0,1}(\Lambda_J^{0,*}(\xi) \otimes \xi, \bar{\Xi}_{J, \gamma, X})$ the cohomology class of the $(0,1)$ -form $H_{J, \gamma, X}$ associated to a DGLA defining couple (γ, X) . The following are equivalent:*

- i) *There exists a DGLA-defining couple (γ, X) such that $H_{J, \gamma, X} = 0$.*
- ii) *$[H_{J, \gamma, X}] = 0$ for every DGLA defining couple (γ, X) .*
- iii) *There exists a DGLA-defining couple (γ, X) such that $[H_{J, \gamma, X}] = 0$.*

Proof. i) \implies ii). Let (γ, X) be a DGLA-defining couple such that $H_{J, \gamma, X} = 0$. Let $(\hat{\gamma}, \hat{X})$ be a DGLA-defining couple, $\hat{\gamma} = e^\lambda \gamma$, $\hat{X} = e^{-\lambda} X + U$, $\lambda \in C^\infty(L)$, $U \in \xi$. By Proposition 4 we have

$$H_{J, \hat{\gamma}, \hat{X}} = \bar{\partial}U - ((\iota_X d\gamma)^{0,1} - \bar{\partial}\lambda) \otimes U.$$

But

$$\bar{\square}_{J,\gamma,X} (e^\lambda U) = e^\lambda \left(\bar{\partial}_J U - \left((\iota_X d\gamma)^{0,1} - \bar{\partial}\lambda \right) \otimes U \right) = e^\lambda H_{J,\hat{\gamma},\hat{X}}$$

and by Proposition 6 it follows that $H_{J,\hat{\gamma},\hat{X}} = \bar{\square}_{J,\hat{\gamma},\hat{X}} U$.

ii) \implies iii) is obvious.

iii) \implies i). Let (γ, X) be a DGLA- defining couple and $U \in \xi$ such that $H_{J,\gamma,X} = \bar{\square}_{J,\gamma,X} U$. Let $(\hat{\gamma}, \hat{X}) = ((\gamma, X - U))$. By Proposition 4

$$H_{J,\hat{\gamma},\hat{X}} = \bar{\square}_{J,\gamma,X} U - \left(\bar{\partial}U - (\iota_X d\gamma)^{0,1} \otimes U \right) = 0.$$

□

Definition 14. Let (ξ, J) be a Levi flat structure. We say that (ξ, J) is exact if it verifies one of the equivalent conditions of Lemma 11.

Example 2. 1) Let (M, J) be a complex manifold and $L = M \times S^1$. Consider the Levi flat structure $\xi = (T(M), J)$ and let (γ, X) a DGLA defining couple, where $\gamma = d\theta$ and $X = \frac{\partial}{\partial\theta}$ where θ is a coordinate on S^1 . Then $H = 0$, so the Levi flat structure is exact.

2) Let (M, J) be a compact complex manifold and $L = \{z \in M : r(z) = 0\}$ a real analytic Levi flat hypersurface in M where r is a real analytic function and $dr \neq 0$ on L . Then the Levi flat structure $(T\mathbb{C}(L), J)$ is exact.

Indeed, let g be a fixed Hermitian metric on M and $Z = \text{grad}_g r / \|\text{grad}_g r\|_g^2$. Then $(\gamma, X) = (d^c r, JZ)$ is a DGLA defining couple for the Levi foliation [3].

Let $p \in L$. There exists holomorphic coordinates $z = (z_1, \dots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$, in a neighborhood U of p such that $L = \{z \in U : x_{2n-1} = 0\}$, so $r = -e^\lambda x_{2n-1}$ with λ a smooth function on U . Set $\text{grad}_g r = \sum_{i=1}^{2n} a_i \frac{\partial}{\partial x_i}$. We have $g(\text{grad}_g r, Y) = dr(Y)$ for every vector field Y , so for every $(\alpha_1, \dots, \alpha_{2n}) \in \mathbb{R}^{2n}$ we have

$$g \left(\sum_{i=1}^{2n} a_i \frac{\partial}{\partial x_i}, \sum_{i=1}^{2n} \alpha_i \frac{\partial}{\partial x_i} \right) = \sum_{i=1}^{2n} \alpha_i \frac{\partial r}{\partial x_i} = - \sum_{i=1}^{2n} \alpha_i e^\lambda \frac{\partial \lambda}{\partial x_i} x_{2n-1} - e^\lambda \alpha_{2n-1}$$

on U . In particular on $U \cap L$ we have $g \left(\sum_{i=1}^{2n} a_i \frac{\partial}{\partial x_i}, \sum_{i=1}^{2n-1} \alpha_i \frac{\partial}{\partial x_i} \right) = 0$ and it follows that $\alpha_1 = \dots = \alpha_{2n-2} = \alpha_{2n} = 0$ on $U \cap L$. So

$$X|_{U \cap L} = J \frac{\text{grad}_g r}{\|\text{grad}_g r\|_g^2} = e^{-\lambda} \frac{\partial}{\partial x_{2n}}.$$

Set now $z_j = x_j + iy_j$, $j = 1, \dots, n$, and let $V = \sum_{j=1}^{n-1} a_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n-1} b_j \frac{\partial}{\partial y_j} \in \ker \gamma$. Then on $U \cap L$ we have

$$\begin{aligned} [V, X] &= \left[\sum_{j=1}^{n-1} a_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n-1} b_j \frac{\partial}{\partial y_j}, e^{-\lambda} \frac{\partial}{\partial y_n} \right] \\ &= -e^{-\lambda} \left(\sum_{j=1}^{n-1} \frac{\partial a_j}{\partial y_n} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n-1} \frac{\partial b_j}{\partial y_n} \frac{\partial}{\partial y_j} \right) + \left(\sum_{j=1}^{n-1} a_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n-1} b_j \frac{\partial}{\partial y_j} \right) (e^{-\lambda}) \frac{\partial}{\partial y_n}, \end{aligned}$$

$$\begin{aligned}
[JV, X] &= \left[-\sum_{j=1}^{n-1} b_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n-1} a_j \frac{\partial}{\partial y_j}, e^{-\lambda} \frac{\partial}{\partial y_n} \right] \\
&= -e^{-\lambda} \left(-\sum_{j=1}^{n-1} \frac{\partial b_j}{\partial y_n} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n-1} \frac{\partial a_j}{\partial y_n} \frac{\partial}{\partial y_j} \right) + \left(-\sum_{j=1}^{n-1} b_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n-1} a_j \frac{\partial}{\partial y_j} \right) (e^{-\lambda}) \frac{\partial}{\partial y_n},
\end{aligned}$$

so

$$[V, X] - \gamma([V, X]) X = -e^{-\lambda} \left(\sum_{j=1}^{n-1} \frac{\partial a_j}{\partial y_n} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n-1} \frac{\partial b_j}{\partial y_n} \frac{\partial}{\partial y_j} \right)$$

and

$$J([JV, X] - \gamma([JV, X]) X) = -e^{-\lambda} \left(\sum_{j=1}^{n-1} -\frac{\partial a_j}{\partial y_n} \frac{\partial}{\partial x_j} - \sum_{j=1}^{n-1} \frac{\partial b_j}{\partial y_n} \frac{\partial}{\partial y_j} \right).$$

It follows that $H = 0$.

8. DEFORMATION THEORY OF LEVI FLAT STRUCTURES

In this paragraph we consider a fixed DGLA defining couple (γ, X) and $T = T_{\gamma, X}$, $H = H_{J, \gamma, X}$ will design the endomorphism and respectively the $(0, 1)$ -form associated to the DGLA defining couple (γ, X) (Definition 12).

Let $\alpha \in \mathcal{Z}^1(L)$ satisfying the Maurer-Cartan equation. We start with a formula which describes the deformation of the Lie bracket:

Lemma 12.

$$[\cdot, \cdot]_{\alpha} = [\cdot, \cdot] + \alpha \wedge T.$$

Proof. We have

$$\begin{aligned}
(\alpha \wedge T)(V, W) &= \alpha(V) T(W) - \alpha(W) T(V) \\
&= \alpha(V) ([W, X] - \gamma([W, X]) X) \\
&\quad - \alpha(W) ([V, X] - \gamma([V, X]) X) \\
&= \alpha(V) [W, X] - \alpha(W) [V, X] \\
&\quad - \alpha(V) \gamma([W, X]) X + \alpha(W) \gamma([V, X]) X.
\end{aligned} \tag{8.1}$$

Since $\gamma(V) = \gamma(W) = 0$ and $\gamma(X) = 1$ it follows that $\gamma([W, X]) = d\gamma(X, W)$ and $\gamma([V, X]) = d\gamma(X, V)$.

Replacing in (8.1) we obtain

$$(\alpha \wedge T)(V, W) = \alpha(V) [W, X] - \alpha(W) [V, X] - (\alpha(V) d\gamma(X, W) - \alpha(W) d\gamma(X, V)) X. \tag{8.2}$$

But $\alpha \in \mathcal{Z}^1(L)$ satisfies the Maurer-Cartan equation, so

$$\delta\alpha + \frac{1}{2} \{\alpha, \alpha\} = d\alpha + \iota_X(d\alpha) \wedge \alpha + \iota_X d\gamma \wedge \alpha - \gamma \wedge \iota_X d\alpha = 0$$

and it follows that

$$d\alpha(V, W) = (-\iota_X(d\alpha) \wedge \alpha - \iota_X d\gamma \wedge \alpha + \gamma \wedge \iota_X d\alpha)(V, W).$$

Taking in account that $\gamma(V) = \gamma(W) = 0$, $\alpha(X) = 0$ we obtain

$$\begin{aligned}
d\alpha(V, W) &= -d\alpha(X, V) \alpha(W) + d\alpha(X, W) \alpha(V) \\
&\quad - d\gamma(X, V) \alpha(W) + d\gamma(X, W) \alpha(V)
\end{aligned}$$

so

$$\alpha(V) d\gamma(X, W) - \alpha(W) d\gamma(X, V) = d\alpha(V, W) - \alpha(V) d\alpha(X, W) + \alpha(W) d\alpha(X, V)$$

and (8.2) becomes

(8.3)

$$(\alpha \wedge T)(V, W) = \alpha(V) [W, X] - \alpha(W) [V, X] - (d\alpha(V, W) - \alpha(V) d\alpha(X, W) + \alpha(W) d\alpha(X, V)) X$$

By (4.1) we have

$$\begin{aligned} [V, W]_\alpha - [V, W] &= \alpha(V) [W, X] - \alpha(W) [V, X] \\ &\quad + (W(\alpha(V)) - V(\alpha(W))) X \\ (8.4) \quad &\quad + \alpha(V) X(\alpha(W)) X - \alpha(W) (X(\alpha(V))) X \\ &\quad - \alpha(V) \alpha([X, W]) X - \alpha(W) \alpha([V, X]) X + \alpha([V, W]) X. \end{aligned}$$

Since $\alpha(X) = 0$ we have

$$\begin{aligned} \alpha(V) X(\alpha(W)) - \alpha(V) \alpha([X, W]) &= \alpha(V) d\alpha(X, W), \\ -\alpha(W) X(\alpha(V)) + \alpha(W) \alpha([X, V]) &= -\alpha(W) d\alpha(X, V) \end{aligned}$$

and

$$W(\alpha(V)) - V(\alpha(W)) - \alpha([W, V]) = -d\alpha(V, W),$$

so (8.4) becomes

$$\begin{aligned} [V, W]_\alpha - [V, W] &= \alpha(V) [W, X] - \alpha(W) [V, X] \\ (8.5) \quad &\quad (\alpha(V) d\alpha(X, W) - \alpha(W) d\alpha(X, V) - d\alpha(V, W)) X. \end{aligned}$$

The Lemma follows now from (8.3) and (8.5). \square

Corollary 4. *Let N_J^α be the Nijenhuis tensor for $(\mathcal{H}(\xi), [\cdot, \cdot]_\alpha, J)$. Then*

$$N_J^\alpha = -4\alpha_J^{0,1} \wedge H.$$

Proof. Let $V, W \in \xi$. By Lemma 12

$$\begin{aligned} [JV, JW]_\alpha &= [JV, JW] + \alpha(JV) T(JW) - \alpha(JW) T(JV) \\ [V, W]_\alpha &= [V, W] + \alpha(V) T(W) - \alpha(W) T(V) \\ J[JV, W]_\alpha &= J[JV, W] + J(\alpha(JV) T(W) - \alpha(W) T(JV)) \\ J[V, JW]_\alpha &= J[V, JW] + J(\alpha(V) T(JW) - \alpha(JW) T(V)) \end{aligned}$$

so

$$\begin{aligned} N_J^\alpha(V, W) &= N_J(V, W) - \alpha(V) (T(W) + JT(JW)) + \alpha(W) (T(V) + JT(JV)) \\ &\quad - J\alpha(JV) (T(W) + JT(JW)) + J\alpha(JW) (T(V) + JT(JV)) \\ &= N_J(V, W) - 2(\alpha(V) HW - \alpha(W) HV + \alpha(JV) JHW - \alpha(JW) JHV). \end{aligned}$$

Since $N_J = 0$ and

$$\begin{aligned} (\alpha^{0,1} \wedge H)(V, W) &= \frac{1}{2}(\alpha(V) + i\alpha(JV)) HW - \frac{1}{2}(\alpha(W) + i\alpha(JW)) HV \\ &= \frac{1}{2}(\alpha(V) HW + \alpha(JV) JHW - \alpha(W) HV - \alpha(JW) JHV), \end{aligned}$$

the Corollary follows. \square

Lemma 13. *Suppose that α is close to 0 and let J_α be a complex structure on ξ_α . Set $\widetilde{J}_\alpha = \omega_\alpha^{-1} J_\alpha \omega_\alpha$ and let S_α be the unique form in $\Lambda_J^{0,1}(\xi) \otimes \xi$ such that the complex structure \widetilde{J}_α on ξ is given by $\widetilde{J}_\alpha = (I + S_\alpha) J (I + S_\alpha)^{-1}$. Then $N_{J_\alpha} = 0$ if and only if $N_{\widetilde{J}_\alpha}^\alpha = 0$.*

Proof. For every $V, W \in \xi_\alpha$ we have

$$N_{\widetilde{J}_\alpha}^\alpha (\omega_\alpha^{-1} V, \omega_\alpha^{-1} W) = N_{J_\alpha} (V, W)$$

and the Lemma follows. \square

From Lemma 3, Lemma 13 Corollary 3 and Lemma 13 we obtain

Corollary 5. *Let J_α a complex structure on ξ_α . We denote S_α the unique form in $\Lambda_J^{0,1}(\xi) \otimes \xi$ such that $\omega_\alpha^{-1} J_\alpha \omega_\alpha = (I + S_\alpha) J (I + S_\alpha)^{-1}$. The following are equivalent:*

- 1) (ξ_α, J_α) is a Levi flat structure.
- 2)

$$(8.6) \quad \delta\alpha + \frac{1}{2} \{\alpha, \alpha\} = 0$$

and

$$(8.7) \quad \overline{\partial}_J^\alpha S_\alpha + \frac{1}{2} [[S_\alpha, S_\alpha]]_\alpha = \frac{1}{4} N_J^\alpha = -\alpha^{0,1} \wedge H$$

where $\overline{\partial}_J^\alpha = \overline{\partial}_{J, [\cdot, \cdot]_\alpha}$ and

$$[[S_\alpha, S_\alpha]]_\alpha = [S_\alpha, S_\alpha]_\alpha - \frac{1}{4} S_\alpha (N_J^\alpha - N_J^\alpha (S_\alpha, S_\alpha)).$$

9. GAUGE EQUIVALENCE

We recall that the group action χ of $\mathcal{G} = \text{Diff}(L)$ on $\mathcal{Z}^1(L)$ is defined in (2.8) such that for $\alpha \in \mathcal{Z}^1(L)$ and $\Phi \in \mathcal{G}$ the distribution ξ_α is integrable if and only if the distribution $\xi_{\chi(\Phi)(\alpha)} = \Phi_* \xi_\alpha$ is integrable (Remark 2). If J_α is a complex structure on ξ_α , by Lemma 7 we define a complex structure

$$J_{\chi(\Phi)(\alpha)} = \omega_{\chi(\Phi)(\alpha)} (I + S_{\chi(\Phi)(\alpha)}) J (I + S_{\chi(\Phi)(\alpha)})^{-1} \omega_{\chi(\Phi)(\alpha)}^{-1}$$

on $\xi_{\chi(\Phi)(\alpha)}$ where

$$(9.1) \quad S_{\chi(\Phi)(\alpha)} = \left(J - \omega_{\chi(\Phi)(\alpha)}^{-1} \Phi_* J_\alpha \Phi_*^{-1} \omega_{\chi(\Phi)(\alpha)} \right) \left(J + \omega_{\chi(\Phi)(\alpha)}^{-1} \Phi_* J_\alpha \Phi_*^{-1} \omega_{\chi(\Phi)(\alpha)} \right)^{-1}.$$

With these definitions we have the following:

Lemma 14. *Let $\Phi \in \mathcal{G}$. Then (ξ_α, J_α) is a Levi flat structure if and only if $(\xi_{\chi(\Phi)(\alpha)}, J_{\chi(\Phi)(\alpha)})$ is a Levi flat structure.*

Proof. By Lemma 13 $N_{J_\alpha} = 0$ if and only if $N_{\widetilde{J}_\alpha}^\alpha = 0$ and $N_{J_{\chi(\Phi)(\alpha)}} = 0$ if and only if $N_{\widetilde{J}_{\chi(\Phi)(\alpha)}}^\alpha = 0$, where $\widetilde{J}_\alpha = \omega_\alpha^{-1} J_\alpha \omega_\alpha$ and $\widetilde{J}_{\chi(\Phi)(\alpha)} = \omega_{\chi(\Phi)(\alpha)}^{-1} J_{\chi(\Phi)(\alpha)} \omega_{\chi(\Phi)(\alpha)}$. But $\omega_\alpha(\xi) = \xi_\alpha = \Phi_* (\xi_{\chi(\Phi)(\alpha)})$ and the Lemma follows. \square

From Lemma 3, Corollary 5 and Lemma 14 we obtain

Corollary 6. *Set*

$$\mathfrak{M}\mathfrak{C}_{\delta, \mathcal{LF}}(L) = \left\{ (\alpha, S) \in \mathcal{Z}^1(L) \times \left(\Lambda_J^{0,1}(\xi) \otimes \xi \right) : \delta a + \frac{1}{2} \{\alpha, \alpha\} = 0, \bar{\partial}_J^\alpha S + \frac{1}{2} [[S, S]]_\alpha = -\alpha_J^{0,1} \wedge H \right\}.$$

Then the moduli space of deformations of Levi flat structures of (ξ, J) is

$$\mathfrak{M}\mathfrak{C}_{\mathcal{LF}}(\xi, J, L) / \sim_{\mathcal{G}}$$

where $(\alpha, S) \sim_{\mathcal{G}} (\beta, Q)$ if there exists $\Phi \in \mathcal{G}$ such that $\beta = \chi(\Phi)(\alpha)$ and $Q = S_\beta$.

Lemma 15. *Let Y be a vector field on L and Φ^Y the flow of Y . Then*

$$\frac{dS_{\chi(\Phi_t^Y)(0)}}{dt} \Big|_{t=0} = -H_Y.$$

where $H_Y \in \Lambda^{0,1}(\xi) \otimes \xi$ is defined in (5.11).

Proof. Let $V \in \xi$. By (9.1)

$$\begin{aligned} & \frac{dS_{\chi(\Phi_t^Y)(0)}}{dt} \Big|_{t=0} (V) \\ &= \frac{d}{dt} \Big|_{t=0} \left(J - \omega_{\chi(\Phi_t^Y)(0)}^{-1} (\Phi_t^Y)_* J (\Phi_t^Y)^{-1} \omega_{\chi(\Phi_t^Y)(0)} \right) \left(J + \omega_{\chi(\Phi_t^Y)(0)}^{-1} (\Phi_t^Y)_* J (\Phi_t^Y)^{-1} \omega_{\chi(\Phi_t^Y)(0)} \right)^{-1} \\ &= \frac{1}{2} \left(\frac{d}{dt} \Big|_{t=0} \omega_{\chi(\Phi_t^Y)(0)}^{-1} (\Phi_t^Y)_* J (\Phi_t^Y)^{-1} \omega_{\chi(\Phi_t^Y)(0)} \right) (JV) \end{aligned} \quad (9.2)$$

Since

$$\frac{d}{dt} \Big|_{t=0} \omega_{\chi(\Phi_t^Y)(0)} = -\frac{d}{dt} \Big|_{t=0} (\chi(\Phi_t^Y)(0)) X,$$

Lemma 4 gives

$$\frac{d}{dt} \Big|_{t=0} \omega_{\chi(\Phi_t^Y)(0)} = \delta(\iota_Y \gamma) X.$$

It follows that

$$\begin{aligned} & \left(\frac{d}{dt} \Big|_{t=0} \left(\omega_{\chi(\Phi_t^Y)(0)}^{-1} (\Phi_t^Y)_* J (\Phi_t^Y)^{-1} \omega_{\chi(\Phi_t^Y)(0)} \right) \right) (V) \\ &= \frac{d}{dt} \Big|_{t=0} \omega_{\chi(\Phi_t^Y)(0)}^{-1} (JV) + \frac{d}{dt} \Big|_{t=0} ((\Phi_t^Y)_*) (JV) + J \left(\frac{d}{dt} \Big|_{t=0} (\Phi_t^Y)^{-1} (V) + \frac{d}{dt} \Big|_{t=0} \omega_{\chi(\Phi_t^Y)(0)} (V) \right) \\ &= -(\delta(\iota_Y \gamma) (JV)) X - \mathcal{L}_Y (JV) + J(\mathcal{L}_Y (V) + (\delta(\iota_Y \gamma) (V)) X) \end{aligned} \quad (9.3)$$

By (2.10) we have

$$\begin{aligned} & \delta(\iota_Y \gamma) (V) = (d\iota_Y \gamma + (\iota_Y \gamma) \iota_X d\gamma - X(\iota_Y \gamma) \gamma) (V) = V(\gamma(Y)) + \gamma(Y) d\gamma(X, V). \end{aligned} \quad (9.4)$$

Since $d\gamma = -\iota_X d\gamma \wedge \gamma$ we have

$$\begin{aligned} V(\gamma(Y)) &= d\gamma(V, Y) + \gamma([V, Y]) = (-\iota_X d\gamma \wedge \gamma)(V, Y) + \gamma([V, Y]) \\ &= -\gamma(Y) (\iota_X d\gamma)(V) + \gamma([V, Y]) = -\gamma(Y) d\gamma(X, V) + \gamma([V, Y]) \end{aligned}$$

and by (9.4) it follows that

$$\delta(\iota_Y \gamma) (V) = \gamma([V, Y]). \quad (9.5)$$

By replacing (9.5) in (9.3), we obtain

$$\begin{aligned} & \left(\frac{d}{dt} \Big|_{t=0} \left(\omega_{\chi(\Phi_t^Y)(0)}^{-1} (\Phi_t^Y)^{-1} J(\Phi_t^Y)_* \omega_{\chi(\Phi_t^Y)(0)} \right) \right) (V) \\ &= -\gamma([JV, Y])X + [JV, Y] + J(-[V, Y] + \gamma([V, Y])X) \end{aligned}$$

We conclude now by (9.2) and (5.11):

$$\frac{dS_{\chi(\Phi_t^Y)(0)}}{dt} \Big|_{t=0} (V) = \frac{1}{2} (-[V, Y] + \gamma([V, Y])X + J(-[JV, Y] + \gamma([JV, Y])X)) = -H_Y.$$

□

Lemma 16. *Let $Y \in \mathcal{H}(L)$. Then*

$$H_Y = \bar{\partial}(Y - \gamma(Y)X) + \gamma(Y)H.$$

Proof. Let $V \in \xi$. We have

$$\begin{aligned} (\bar{\partial}(Y - \gamma(Y)X))(V) &= \frac{1}{2} ([V, Y - \gamma(Y)X] + J[JV, Y - \gamma(Y)X]) \\ &= \frac{1}{2} ([V, Y] - \gamma(Y)[V, X] - V(\gamma(Y))X) \\ &\quad + \frac{1}{2} J([JV, Y] - \gamma(Y)[JV, X] - (JV)(\gamma(Y))X) \end{aligned} \tag{9.6}$$

But

$$d\gamma(V, Y) = V(\gamma(Y)) - \gamma[V, Y]$$

and by Lemma 2

$$d\gamma(V, Y) = (-\iota_X d\gamma \wedge \gamma)(V, Y) = -\gamma(Y)(\iota_X d\gamma)(V) = -\gamma(Y)d\gamma(X, V) = -\gamma(Y)\gamma([V, X]),$$

so

$$V(\gamma(Y)) = \gamma[V, Y] - \gamma(Y)\gamma([V, X]) \tag{9.7}$$

and similarly

$$(JV)(\gamma(Y)) = \gamma([JV, Y]) - \gamma(Y)\gamma([JV, X]). \tag{9.8}$$

It follows that

$$\gamma(Y)H(V) = \frac{1}{2} (\gamma(Y)[V, X] - \gamma(Y)\gamma([V, X])X + J(\gamma(Y)[JV, X] - \gamma(Y)\gamma([JV, X])X)). \tag{9.9}$$

and by (9.7) and (9.8) in the addition of (9.6) and (9.9) it follows that

$$(\bar{\partial}(Y - \gamma(Y)X))(V) + \gamma(Y)H(V) = \frac{1}{2} ([V, Y] - \gamma([V, Y])X) + \frac{1}{2} J([JV, Y] - \gamma([JV, Y])X) = H_Y(V)$$

and the Lemma is proved. □

Corollary 7.

$$\bar{\partial}H_Y = (\delta(\gamma(Y)))^{0,1} \wedge H$$

Proof.

$$\begin{aligned} \bar{\partial}H_Y &= \bar{\partial}(\gamma(Y)H) = \bar{\partial}(\gamma(Y)) \wedge H + \gamma(Y)\bar{\partial}H \\ &= \bar{\partial}(\gamma(Y)) \wedge H + \gamma(Y)(\iota_X d\gamma)^{0,1} \wedge H \\ &= (d\gamma(Y) + \gamma(Y)\iota_X d\gamma)^{0,1} \wedge H. \end{aligned}$$

Since

$$\delta(\gamma(Y)) = d\gamma(Y)\gamma(Y) + \gamma(Y)\iota_X d\gamma - X(\gamma(Y))\gamma$$

it follows that

$$\delta(\gamma(Y)) = d\gamma(Y)\gamma(Y) + \gamma(Y)\iota_X d\gamma$$

on ξ and the Corollary follows. \square

Corollary 8. *Let $\beta \in \mathcal{Z}^1(L)$ and $\varphi \in C^\infty(L)$. Then*

$$(\beta + \delta\varphi)^{0,1} \wedge H = \beta^{0,1} \wedge H + \bar{\partial}(\varphi H).$$

In particular the map $\Phi_H : \beta \mapsto \beta^{0,1} \wedge H$ induces an application

$$(9.10) \quad \Phi_H : H^1(\mathcal{Z}(L), \delta) \rightarrow H^2(\Lambda^{0,*}(\xi) \otimes \xi, \bar{\partial}).$$

Proof.

$$(\beta + \delta\varphi)^{0,1} \wedge H = \beta^{0,1} \wedge H + (\delta\varphi)^{0,1} \wedge H = \beta^{0,1} \wedge H + (d\varphi + \varphi\iota_X d\gamma - d\varphi(X)\gamma)^{0,1} \wedge H.$$

Since $\gamma^{0,1} = 0$, by using Lemma 10 we obtain

$$(\beta + \delta\varphi)^{0,1} \wedge H = \beta^{0,1} \wedge H + \bar{\partial}\varphi \wedge H + \varphi\bar{\partial}H = \beta^{0,1} \wedge H + \bar{\partial}(\varphi H).$$

\square

10. MODULI SPACE OF DEFORMATIONS OF LEVI FLAT STRUCTURES AND RIGIDITY

Definition 15. *Let L be a smooth manifold, (ξ, J) a Levi flat structure on L , and I an open interval in \mathbb{R} containing the origin. A deformation of the Levi flat structure (ξ, J) is a smooth family $\{(\xi_t, J_t)\}_{t \in I}$ of Levi flat structures on L such that $(\xi_0, J_0) = (\xi, J)$.*

Remark 8. *By Corollary 5 a deformation of the Levi flat structure (ξ, J) is given by a family $\{(\alpha_t, S_{\alpha_t})\}_{t \in I}$, $\alpha_t \in \mathcal{Z}^1(L)$, $S_{\alpha_t} \in \Lambda_J^{0,1}(\xi) \otimes \xi$ such that α_t verifies (8.6) and S_{α_t} verifies (8.7) for every $t \in I$.*

Definition 16. *A $\mathfrak{MC}_{\delta, \mathcal{LF}}(L)$ -valued curve through the origin is a smooth mapping $\lambda : [-a, a] \rightarrow \mathfrak{MC}_{\delta, \mathcal{LF}}(L)$, $a > 0$, such that $\lambda(0) = 0$. We say that $(\beta, P) \in \mathcal{Z}^1(L) \times (\Lambda^{0,1}(\xi) \otimes \xi)$ is the tangent vector at the origin to the $\mathfrak{MC}_{\delta, \mathcal{LF}}(L)$ -valued curve λ through the origin if $(\beta, P) = \lim_{t \rightarrow 0} \frac{\lambda(t)}{t} = \frac{d\lambda}{dt}|_{t=0}$.*

Theorem 1. *Let L be a smooth manifold and (ξ, J) a Levi flat structure on L . Let $\{(\xi_t, J_t)\}_{t \in I}$ be a deformation of (ξ, J) given by $\{(\alpha_t, S_{\alpha_t})\}_{t \in I}$, $\alpha_t = t\beta + o(t)$, $S_{\alpha_t} = tP + o(t)$, $\alpha_t, \beta \in \mathcal{Z}^1(L)$, $S_{\alpha_t}, P \in \Lambda_J^{0,1}(\xi) \otimes \xi$. Then:*

1)

$$(10.1) \quad \delta\beta = 0$$

$$(10.2) \quad \bar{\partial}P = -\beta^{0,1} \wedge H.$$

2) *Let $\beta' \in \mathcal{Z}^1(L)$, $P' \in \Lambda^{0,1}(\xi) \otimes \xi$ such that β' verifies (10.1) and P' verifies (10.2). Then $(\beta, P) \sim_{\mathcal{G}} (\beta', P')$ if and only if there exists $Y \in \mathcal{H}(L)$ such that*

$$(10.3) \quad \beta - \beta' = \delta\iota_Y \gamma$$

$$(10.4) \quad P - P' = -H_Y.$$

Proof. 1) follows from (8.6) and (8.7) and 2) from (2.11) and Lemma 15. \square

Definition 17. Let $\mathfrak{Z}^p(L, \xi) = \mathcal{Z}^p(L) \oplus \Lambda^{0,p}(\xi) \otimes \xi$, $\mathfrak{Z} = \bigoplus_{p \in \mathbb{N}} \mathfrak{Z}^p$ and $\mathfrak{d} = (\mathfrak{d}^p)_{p \in \mathbb{N}} : \mathfrak{Z} \rightarrow \mathfrak{Z}$, where $\mathfrak{d}^p : \mathfrak{Z}^p \rightarrow \mathfrak{Z}^{p+1}$,

$$(10.5) \quad \mathfrak{d}^p(\alpha, P) = \left(\delta^p \alpha, \bar{\partial} P + (-1)^{p+1} \alpha^{0,p} \wedge H \right).$$

Proposition 7. $\mathfrak{d} \circ \mathfrak{d} = 0$.

Proof. Let $(\alpha, P) \in \mathfrak{Z}^p(L, \xi)$. By Corollary 1. we have $\delta^{p+1} \delta^p = 0$ so by Lemma 10 it follows that

$$\begin{aligned} \mathfrak{d}^{p+1} \mathfrak{d}^p(\alpha, P) &= \mathfrak{d}^{p+1} \left(\delta^p \alpha, \bar{\partial} P + (-1)^{p+1} \alpha^{0,p} \wedge H \right) \\ &= \left(0, \left((-1)^{p+1} \bar{\partial} (\alpha^{0,p} \wedge H) + (-1)^{p+2} (\delta^p \alpha)^{0,p+1} \wedge H \right) \right) \\ &= (-1)^{p+1} \left(0, \bar{\partial} \alpha^{0,p} \wedge H + (-1)^p \alpha^{0,p} \wedge \bar{\partial} H - (d\alpha + \{\gamma, \alpha\})^{0,p+1} \wedge H \right) \\ &= (-1)^{p+1} \left(0, \bar{\partial} \alpha^{0,p} \wedge H + (-1)^p \alpha^{0,p} \wedge \bar{\partial} H - \bar{\partial} \alpha^{0,p} \wedge H + \{\gamma, \alpha\}^{0,p+1} \wedge H \right) \\ &= (-1)^{p+1} \left(0, (-1)^p \alpha^{0,p} \wedge (\iota_X d\gamma)^{0,1} \wedge H - (\iota_X d\gamma \wedge \alpha - \gamma \wedge \iota_X d\alpha)^{0,p+1} \right) \wedge H \\ &= (-1)^{p+1} \left(0, (-1)^p \alpha^{0,p} \wedge (\iota_X d\gamma)^{0,1} - (\iota_X d\gamma)^{0,1} \wedge \alpha^{0,p} - \gamma^{0,1} \wedge (\iota_X d\alpha)^{0,p} \right) \wedge H. \end{aligned}$$

But $\gamma^{0,1} = 0$ and the Proposition is proved. \square

Notation 4. We denote \mathcal{A} the set of $(\beta, P) \in \mathfrak{Z}^1(L, \xi)$ such that β verifies (10.1) and P verifies (10.2).

Theorem 2. Let L be a smooth manifold and (ξ, J) a Levi flat structure on L . Then $\mathcal{A} / \sim_{\mathcal{G}}$ is canonically isomorphic with $H^1(\mathfrak{Z}, \mathfrak{d})$.

Proof. Let $[(\beta, P)] \in \mathcal{A} / \sim_{\mathcal{G}}$, $(\beta, P) \in \mathcal{A}$. By (10.1) and (10.2), (β, P) defines an element $\widehat{(\beta, P)} \in H^1(\mathfrak{Z}, \mathfrak{d})$.

Suppose that $[(\beta, P)] = [(0, 0)]$. By (10.3) and (10.4) there exists $Y \in \mathcal{H}(L)$ such that $\beta = \delta(\gamma(Y))$ and $P = -H_Y$. The Lemma 16 implies that

$$P = \bar{\partial}(Y - \gamma(Y)X) + \gamma(Y)H$$

so

$$\mathfrak{d}^0(\gamma(Y), -(Y - \gamma(Y)X)) = (\delta(\gamma(Y)), -\bar{\partial}(Y - \gamma(Y)X) + \gamma(Y)H) = (\beta, -H_Y) = (\beta, P).$$

It follows that $\widehat{(\beta, P)} = \widehat{(0, 0)}$ and the map $F : \mathcal{A} \rightarrow H^1(\mathfrak{Z}, \mathfrak{d})$, $F([(\beta, P)]) = \widehat{(\beta, P)}$, is well defined.

Let now $\widehat{(\beta, P)} \in H^1(\mathfrak{Z}, \mathfrak{d})$, with $(\beta, P) \in \mathfrak{Z}^1(L, \xi)$, $\mathfrak{d}^1(\beta, P) = (\delta\beta, \bar{\partial}P + \beta^{0,1} \wedge H) = (0, 0)$. It follows that β verifies (10.1) and P verifies (10.2). In particular $(\beta, P) \in \mathcal{A}$.

Suppose now that $\widehat{(\beta, P)} = \widehat{(0, 0)}$, i.e. $(\beta, P) = \mathfrak{d}^0(0, V) = (0, -\bar{\partial}V)$, $V \in \xi$. Since $\bar{\partial}V = H_V$ by Remark 5 ii), it follows that $P = -H_V$. By (10.3) and (10.4) it follows that $[(\beta, P)] = [0, 0] \in \mathcal{A} / \sim$. In particular F is an isomorphism. \square

The Theorems 1 and 2 justify the following:

Definition 18. The infinitesimal deformations of the Levi flat structure (ξ, J) is the collection of cohomology classes in $H^1(\mathfrak{Z}, \mathfrak{d})$ of the tangent vectors at 0 to $\mathfrak{MC}_{\delta, \mathcal{LF}}(L)$ -valued curves. We denote by $T_{[0]}(\mathfrak{MC}_{\delta, \mathcal{LF}}(L) / \sim_{\mathcal{G}})$ the set of infinitesimal deformations of (ξ, J) .

Definition 19. Let L be a smooth manifold, (ξ, J) a Levi flat structure on L . We say that (L, ξ, J) is infinitesimally rigid (respectively strongly infinitesimally rigid), if for any smooth family $\{(\alpha_t, S_{\alpha_t})\}_{t \in I}$ defining a deformation of the Levi flat structure (ξ, J) , the class of the tangent vector to the $\mathfrak{MC}_{\delta, \mathcal{LF}}(L)$ -valued curve $t \mapsto (\alpha_t, S_{\alpha_t})$ in $\mathcal{A}/\sim_{\mathcal{G}}$ vanishes (respectively the tangent vector to the $\mathfrak{MC}_{\delta, \mathcal{LF}}(L)$ -valued curve $t \mapsto (\alpha_t, S_{\alpha_t})$ vanishes).

Corollary 9. Let L be a smooth manifold (ξ, J) a Levi flat structure on L such that $H^1(\mathfrak{Z}, \mathfrak{d}) = 0$. Then (L, ξ, J) is infinitesimally rigid.

Corollary 10. Let L be a smooth manifold and (ξ, J) an exact Levi flat structure on L . Then $\mathcal{A}/\sim_{\mathcal{G}}$ is canonically isomorphic to $H^1(\mathcal{Z}^*(L), \delta) \times H^1(\Lambda_J^{0,*}(\xi) \otimes \xi)$ and we have a canonical injection

$$T_{[0]}(\mathfrak{MC}_{\delta, \mathcal{LF}}(L) / \sim) \hookrightarrow H^1(\mathcal{Z}^*(L), \delta) \times H^1(\Lambda_J^{0,*}(\xi) \otimes \xi, \bar{\partial}_J).$$

Proof. By Theorem 2 $\mathcal{A}/\sim_{\mathcal{G}}$ is canonically isomorphic to $H^1(\mathfrak{Z}, \mathfrak{d})$. There exists a defining couple (γ, X) such that the $(0, 1)$ -form H associated to (γ, X) vanishes. It follows that $\mathfrak{d} = \delta \oplus \bar{\partial}_J$ and so

$$H^1(\mathfrak{Z}, \mathfrak{d}) = H^1(\mathcal{Z}^*(L), \delta) \times H^1(\Lambda_J^{0,*}(\xi) \otimes \xi, \bar{\partial}_J).$$

□

Example 3. Let $L = \mathbb{CP}_n \times S^1$ with its exact Levi flat structure given by the DGLA defining couple $(\gamma, X) = (dt, \frac{\partial}{\partial t})$, where t runs in S^1 . Since $\delta = d_b$ and \mathbb{CP}_n is simply connected and infinitesimally rigid, it follows that $H^1(\mathfrak{Z}, \mathfrak{d}) = 0$.

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